Vektorprodukt für einen 3-dim eukl. VR V

↳ Abbildung "... × ...": $V \times V \longrightarrow V$

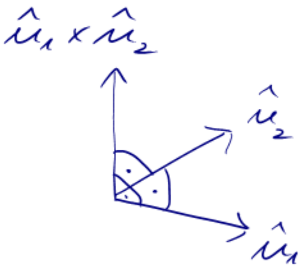
(auch: Kreuzprodukt) $\qquad \vec{a}, \vec{b} \longmapsto \vec{a} \times \vec{b}$

mit definierenden Eigenschaften

(1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (Antisymmetrie)

(2) $\vec{a} \times (\vec{b} + \lambda \vec{c}) = \vec{a} \times \vec{b} + \lambda\, \vec{a} \times \vec{c}$ (Linearität)

(3) $\hat{u}_1, \hat{u}_2$ orthonormal $\Longrightarrow$ $\hat{u}_1, \hat{u}_2, \hat{u}_1 \times \hat{u}_2$ rechtshändige ONB



(Orthonormalität)

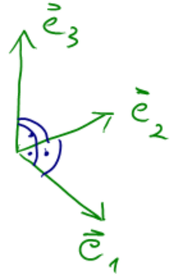
aus (1) und (2) folgt:

$$\boxed{\vec{a} \parallel \vec{b} \quad \Rightarrow \quad \vec{a} \times \vec{b} = \vec{0}}$$

[denn: • $\vec{a} \times \vec{a} \overset{(1)}{=} -\vec{a} \times \vec{a} \quad \rightarrow \quad \vec{a} \times \vec{a} = \vec{0}$;

• $\vec{a} \parallel \vec{b}$ bedeutet $\vec{b} = \lambda \vec{a} \quad \rightarrow \quad \vec{a} \times \vec{b} = \vec{a} \times (\lambda \vec{a}) \overset{(2)}{=} \lambda\, \vec{a} \times \vec{a} = \lambda \vec{0} = \vec{0}$.]

Bestimmung von $\vec{a} \times \vec{b}$ in Komponenten bzgl. rechtshändigen ONB $B = \{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \}$:



$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_B = \sum_{\ell=1}^{3} a_\ell \vec{e}_\ell \quad ; \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_B = \sum_{m=1}^{3} b_m \vec{e}_m$$

$$\to \quad \vec{a} \times \vec{b} = \left(\sum_{l} a_l \vec{e}_l\right) \times \left(\sum_{m} b_m \vec{e}_m\right) \overset{(2)}{=} \sum_{l,m=1}^{3} a_l b_m \; \vec{e}_l \times \vec{e}_m$$

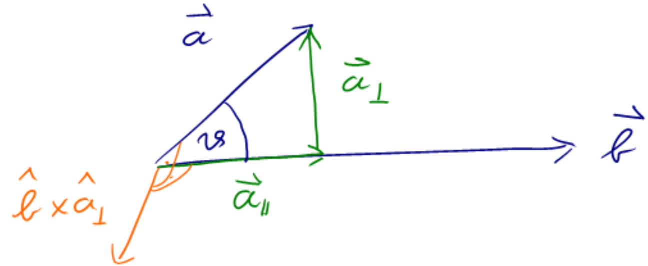
$$\overset{!}{=} \sum_{\substack{l,m=1,\\ l<m}}^{3} \left(a_l b_m \vec{e}_l \times \vec{e}_m + a_m b_l \underbrace{\vec{e}_m \times \vec{e}_l}_{\overset{(1)}{=} \ominus \vec{e}_l \times \vec{e}_m} \right)$$

$$= \sum_{\substack{l,m=1,\\ l<m}}^{3} \left(a_l b_m - a_m b_l \right) \vec{e}_l \times \vec{e}_m$$

$$= (a_1 b_2 - a_2 b_1) \underbrace{\vec{e}_1 \times \vec{e}_2}_{= \vec{e}_3} + (a_1 b_3 - a_3 b_1) \underbrace{\vec{e}_1 \times \vec{e}_3}_{= -\vec{e}_2} + (a_2 b_3 - a_3 b_2) \underbrace{\vec{e}_2 \times \vec{e}_3}_{= \vec{e}_1}$$

$$\to \quad \boxed{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_B \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_B = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}_B}$$

Orthogonalkomponente mittels Vektorprodukt



bisher:

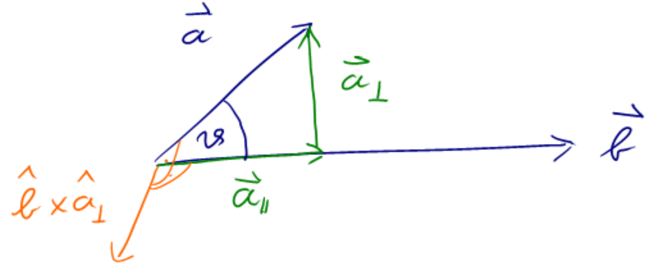
- $\vec{a}_\parallel = \langle \hat{b}, \vec{a} \rangle \, \hat{b}$,
- $\vec{a}_\perp = \vec{a} - \vec{a}_\parallel$,
- $\langle \vec{a}, \vec{b} \rangle = |\vec{a}| \, |\vec{b}| \cos\vartheta$

wir zeigen:

a) $\vec{a}_\perp = (\hat{b} \times \vec{a}) \times \hat{b}$

b) $|\vec{a} \times \vec{b}| = |\vec{a}| \, |\vec{b}| \, |\sin\vartheta|$

zu a):



$$(\hat b \times \vec a) = \hat b \times (\vec a_\parallel + \vec a_\perp)$$

$$= \underbrace{\hat b \times \vec a_\parallel}_{=\vec 0,\ \text{da } \hat b \parallel \vec a_\parallel} + \hat b \times \vec a_\perp$$

$$= \hat b \times (|\vec a_\perp|\, \hat a_\perp) = |\vec a_\perp|\, \hat b \times \hat a_\perp$$

wegen $(\hat b \times \hat a_\perp) \times \hat b = \hat a_\perp$ also $(\vec b \times \vec a) \times \hat b = |\vec a_\perp|\, \hat a_\perp = \vec a_\perp$.

zu b): $|\vec a_\perp|^2 = |\vec a|^2 - |\vec a_\parallel|^2 = |\vec a|^2 (1 - \cos^2\vartheta) = |\vec a|^2 \sin^2\vartheta$

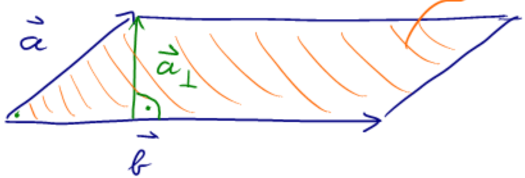
$$|\vec a_\parallel|^2 = |\langle \hat b, \vec a\rangle|^2 = |\vec a|^2 \cos^2\vartheta$$

d.h. $|\vec a|\,|\sin\vartheta| = |\vec a_\perp| \overset{a)}{=} |\hat b \times \vec a| \qquad | \cdot |\vec b|$

$$|\vec b \times \vec a| = |\vec a|\,|\vec b|\,|\sin\vartheta| .$$

Anwendungen des Vektorprodukts:

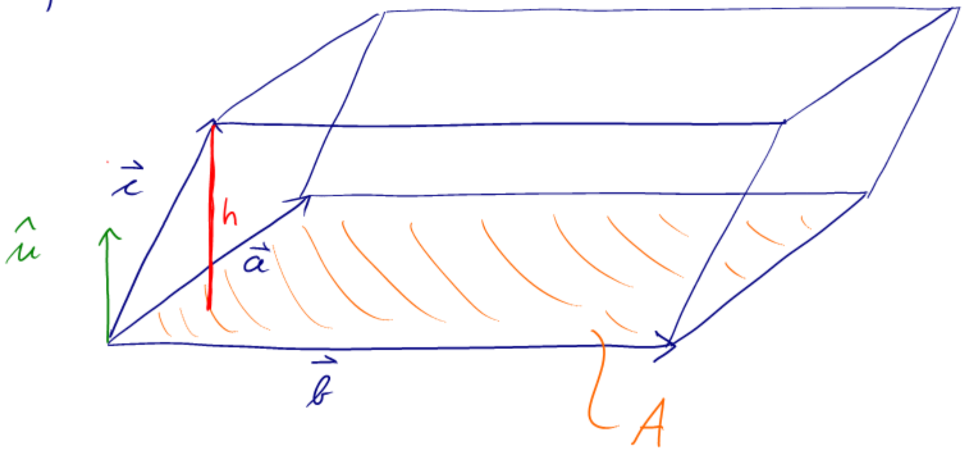
- Flächeninhalt eines Parallelogramms mit Kanten $\vec{a}, \vec{b}$:



$$A \stackrel{!}{=} |\vec{a} \times \vec{b}|$$

$$A = |\vec{a}_\perp| \cdot |\vec{b}| = |\vec{a} \times \hat{b}| \cdot |\vec{b}| = |\vec{a} \times \vec{b}|$$

- Volumeninhalt eines <u>Spats</u> (auch: <u>Parallelepipeds</u>) mit Kanten $\vec{a}$, $\vec{b}$, $\vec{c}$



$$\hat{u} = \frac{\vec{b} \times \vec{a}}{|\vec{b} \times \vec{a}|} \quad , \quad A = |\vec{b} \times \vec{a}| \quad , \quad h = |\langle \hat{u}, \vec{c} \rangle|$$

$$\to \quad V = hA = \overbrace{|\langle \hat{u}, \hat{c} \rangle|}^{h} \, \overbrace{|\vec{b} \times \vec{a}|}^{A} = \left\langle \frac{\overbrace{\vec{b} \times \vec{a}}^{\vec{u}}}{\cancel{|\vec{b} \times \vec{a}|}}, \vec{c} \right\rangle \cdot \cancel{|\vec{b} \times \vec{a}|}$$

$$\to \quad \boxed{V = |\langle \vec{a} \times \vec{b}, \vec{c} \rangle|} \leftarrow$$

<u>Spatprodukt</u> der Vektoren $\vec{a}$, $\vec{b}$ und $\vec{c}$