letzte Vrlsg.: Ableitung vektor-wertiger Fktn: $\vec{f}: D \to V$

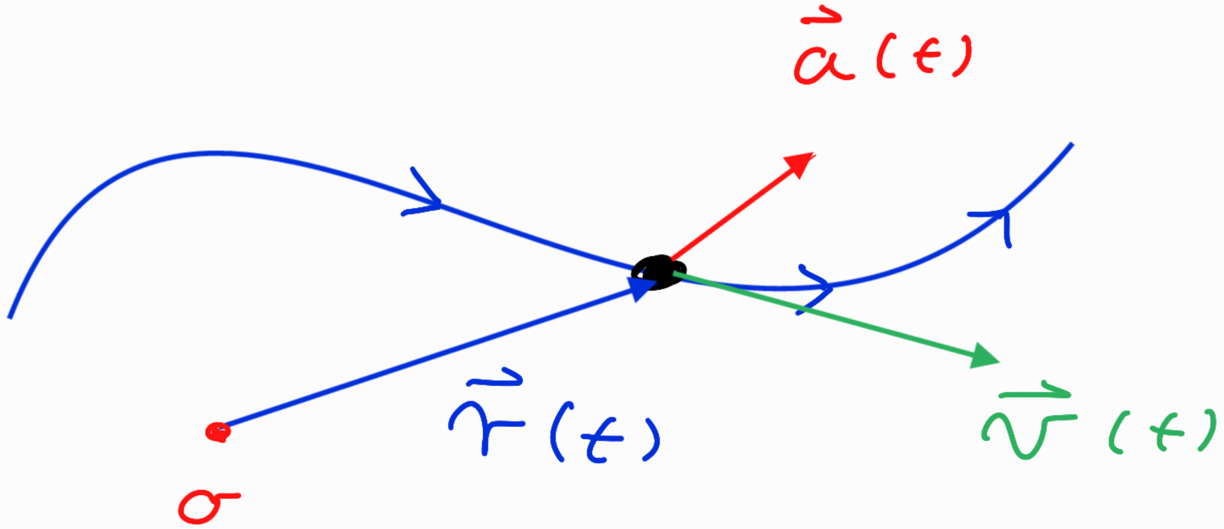
- $\vec{f}'(x) = \lim_{h \to 0} \frac{1}{h}\left(\vec{f}(x+h) - \vec{f}(x)\right)$

- analog: $\frac{\partial \vec{f}}{\partial x_e}(x)$

- $\vec{f}(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}_B$

$\Rightarrow \vec{f}'(x) = \begin{pmatrix} f_1'(x) \\ \vdots \\ f_n'(x) \end{pmatrix}_B$

- momentane <u>Geschwindigkeit</u> und <u>Beschleunigung</u>:



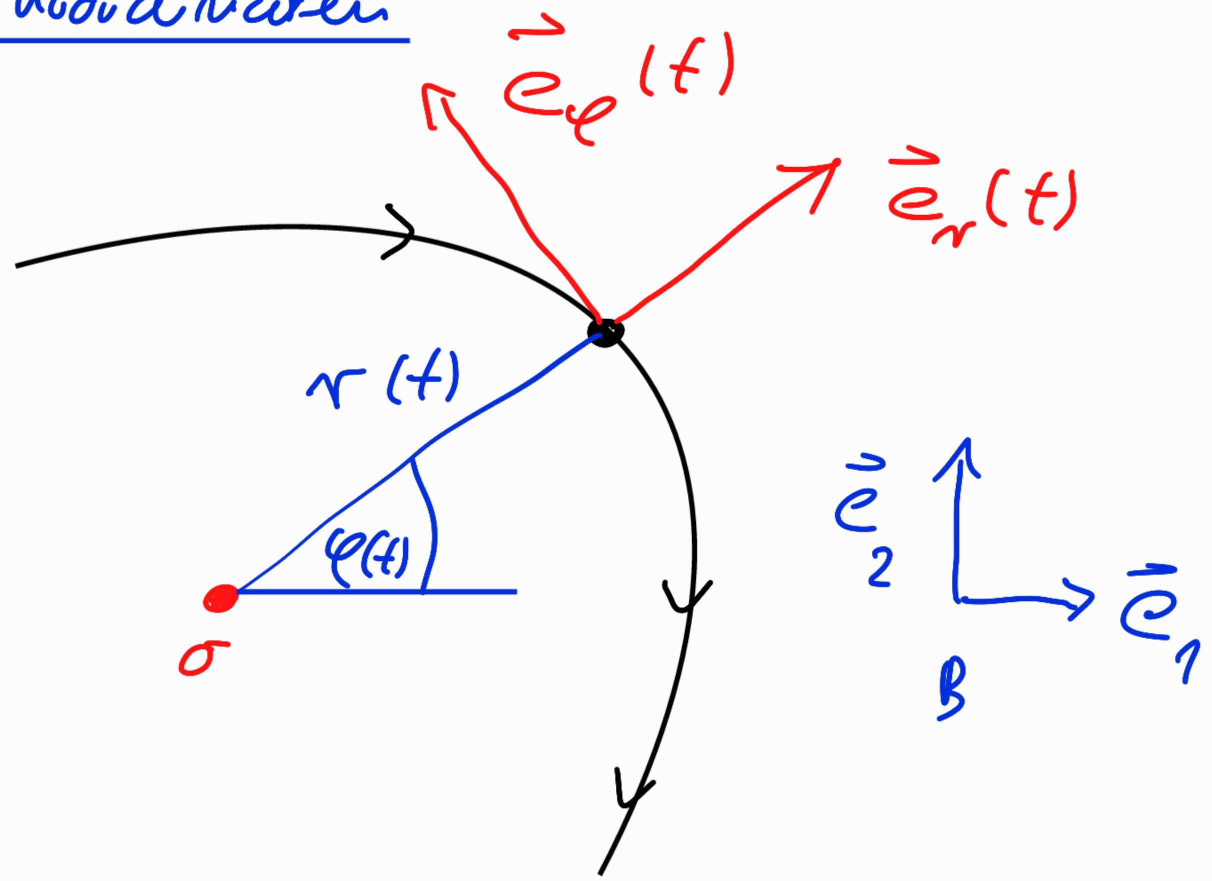
Bahn: $t \mapsto \vec{r}(t)$

$\Rightarrow$ • $\vec{v}(t) = \dfrac{d\vec{r}(t)}{dt} = \dot{\vec{r}}(t)$

• $\vec{a}(t) = \dfrac{d\vec{v}}{dt}(t) = \dot{\vec{v}}(t)$

$= \ddot{\vec{r}}(t)$

Geschwindigkeit und Beschleunigung in Polarkoordinaten



zeitabhängige Koord. $r(t)$, $\varphi(t)$

" " ONB

$$\vec{e}_r(t) = \begin{pmatrix} \cos\varphi(t) \\ \sin\varphi(t) \end{pmatrix}_B, \quad \vec{e}_\varphi(t) = \begin{pmatrix} -\sin\varphi(t) \\ \cos\varphi(t) \end{pmatrix}_B$$

$$\vec{r}(t) = r(t)\,\vec{e}_r(t)$$

$$\rightarrow \quad \vec{v}(t) = \dot{\vec{r}}(t) = \dot{r}(t)\vec{e}_r(t) + r(t)\dot{\vec{e}}_r(t)$$

$$\vec{a}(t) = \dot{\vec{v}}(t) = \frac{d}{dt}\left(\dot{r}\vec{e}_r + r\dot{\vec{e}}_r\right)$$

$$\vec{a}(t) = \ddot{r}\vec{e}_r + 2\dot{r}\dot{\vec{e}}_r + r\ddot{\vec{e}}_r$$

$$\dot{\vec{e}}_r = ? \quad , \quad \ddot{\vec{e}}_r = ? \quad , \quad \dot{\vec{e}}_\varphi = ?$$

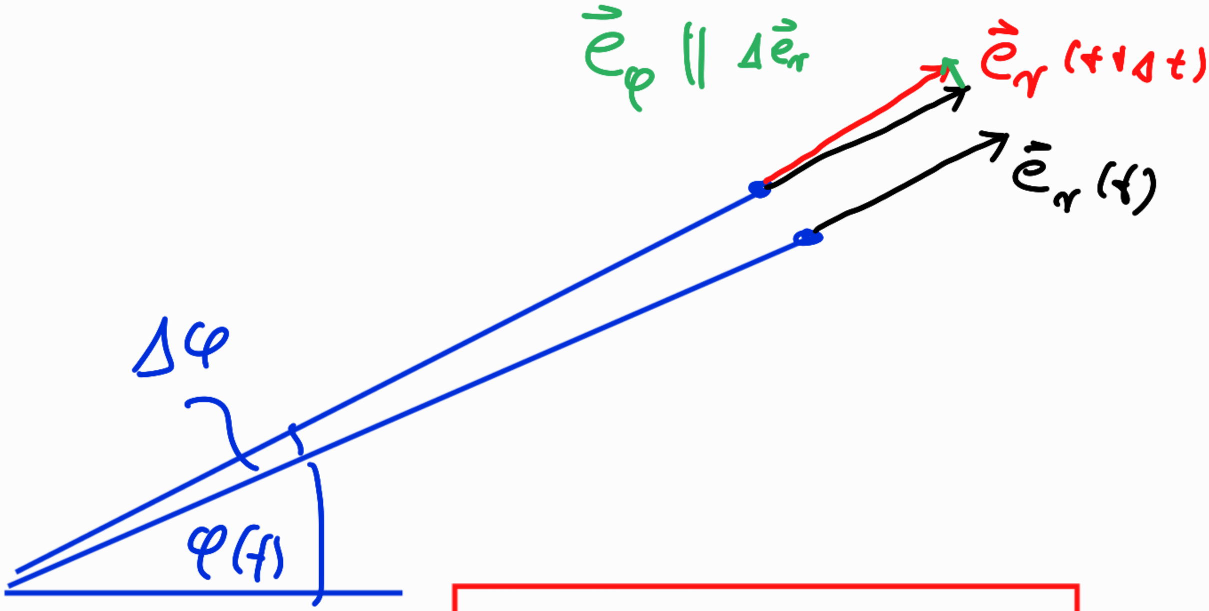
$$\dot{\vec{e}}_r = \frac{d}{dt}\begin{pmatrix} \cos\varphi(t) \\ \sin\varphi(t) \end{pmatrix}_B = \begin{pmatrix} -\dot{\varphi}\sin\varphi \\ \dot{\varphi}\cos\varphi \end{pmatrix}_B$$

$$= \dot{\varphi}\underbrace{\begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}_B}_{=\vec{e}_\varphi}$$

$$\rightarrow \quad \boxed{\dot{\vec{e}}_r = \dot{\varphi}\,\vec{e}_\varphi}$$

analog:

$$\boxed{\dot{\vec e}_\varphi = -\underline{\dot\varphi}\,\vec e_r}$$



$$\boxed{\dot{\vec e}_r = \dot\varphi\,\vec e_\varphi}$$

$$\vec v = \dot r\,\vec e_r + r\,\dot{\vec e}_r$$

$$\Rightarrow \boxed{\vec v = \underbrace{\dot r}_{\text{Radial-}}\,\vec e_r + \underbrace{r\,\dot\varphi}_{\text{Azimuthalgeschw.}}\,\vec e_\varphi}$$

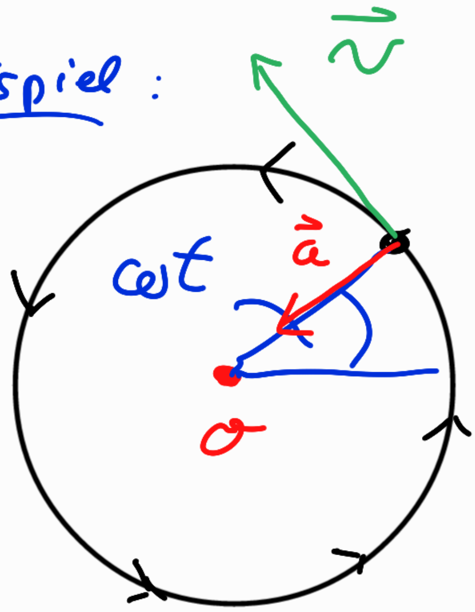
$$\vec{a} = \ddot{r}\vec{e}_r + 2\dot{r}\dot{\vec{e}}_r + r\ddot{\vec{e}}_r$$

$$= \ddot{r}\vec{e}_r + 2\dot{r}\dot{\varphi}\vec{e}_\varphi + r\ddot{\vec{e}}_r$$

$$\left[\ddot{\vec{e}}_r = \frac{d}{dt}(\dot{\varphi}\vec{e}_\varphi) = \ddot{\varphi}\vec{e}_\varphi - \dot{\varphi}^2\vec{e}_r \right]$$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\varphi}^2)\vec{e}_r + (2\dot{r}\dot{\varphi} + \ddot{\varphi})\vec{e}_\varphi}$$

Beispiel:



gleichmäßige Kreis-bewegung:

$$r(t) = R$$

$$\varphi(t) = \omega t$$

$$\rightsquigarrow \vec{v} = R\omega\vec{e}_\varphi$$

$$\vec{a} = -R\omega^2\vec{e}_r$$

<u>Integral</u> einer Fkt. f in den Grenzen a und b:

$b \leftarrow$ obere Grenze

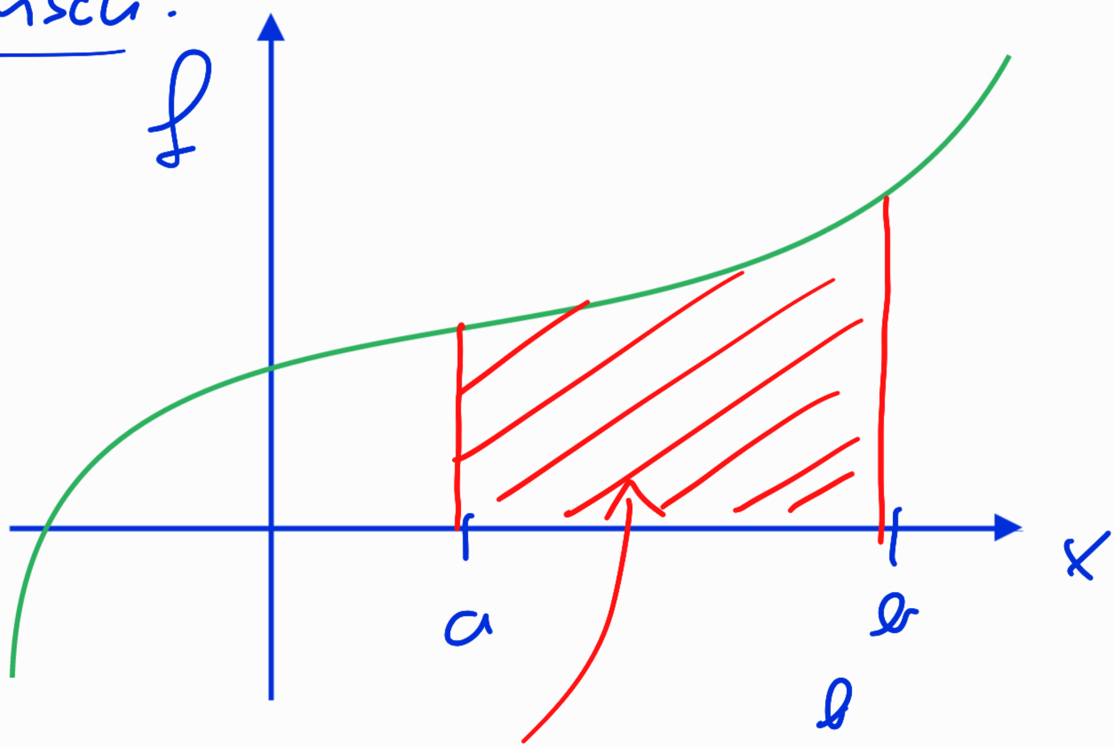
$$\int_a^b f(x)\,dx$$

Integrand

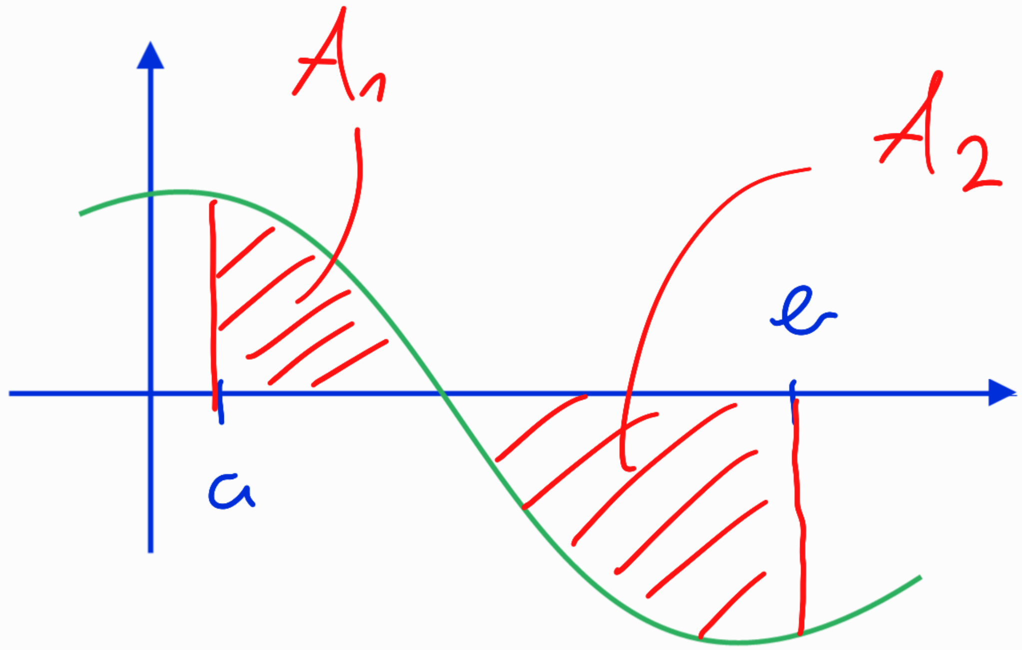
Integrationsvariable $(x, \tilde{x}, s, t, u, \ldots)$

untere Grenze

<u>geometrisch</u>:

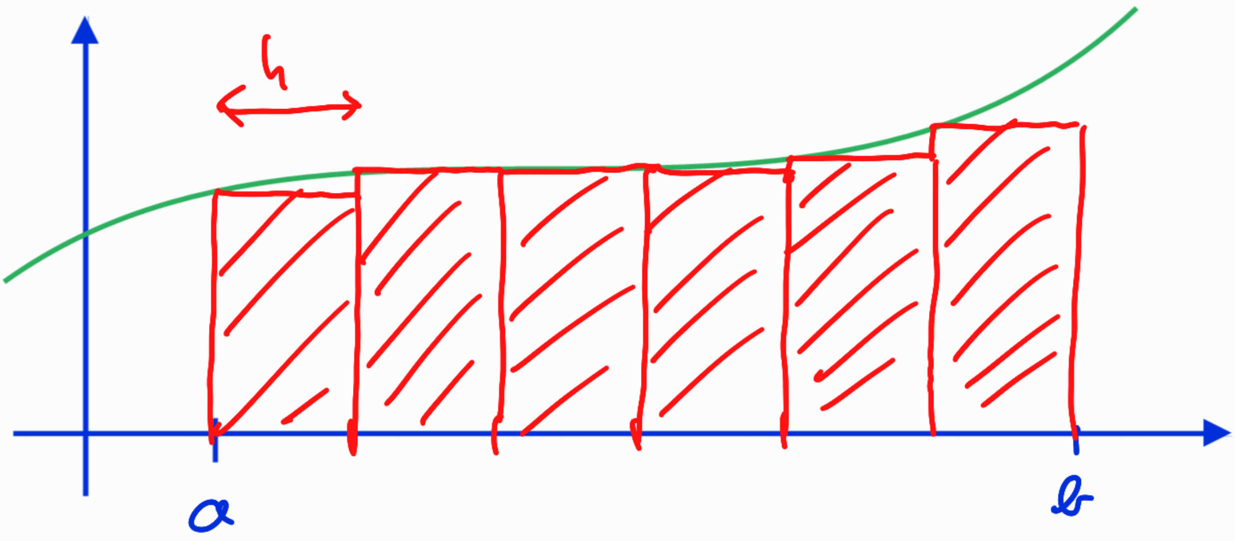


Flächeninhalt $= \int_a^b f(x)\,dx$



$$\int_a^b f(x)\,dx = A_1 - A_2$$

genauer:



$$\boxed{\int_a^b f(x)\,dx := \lim_{h\to 0} \sum_{l=0}^{(b-a)/h} h\, f(a+lh)}$$

→ Eigenschaften:

1) $$\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$$

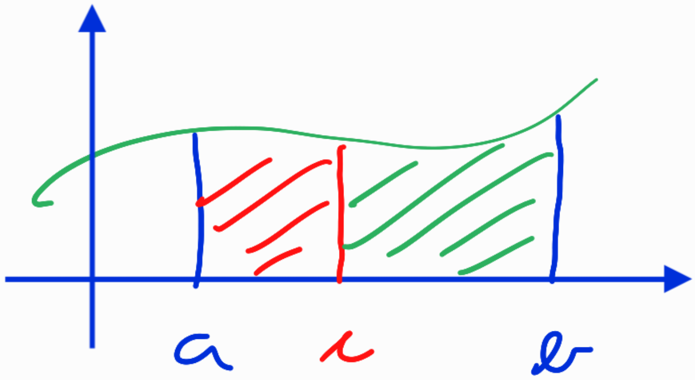
$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(Linearität)

<u>Def</u>: Falls $b < a$:

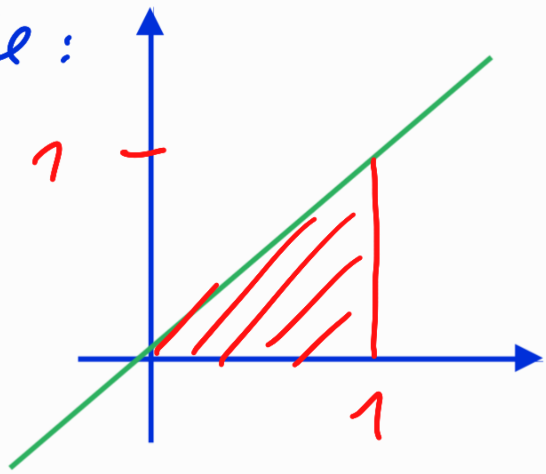
$$\boxed{\int_a^b f(x)\,dx := -\int_b^a f(x)\,dx}$$!

2) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

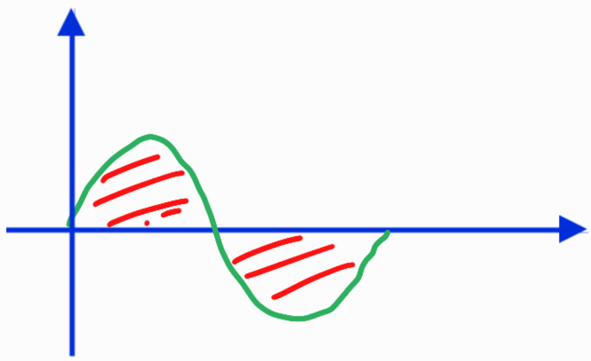


elementare Beispiele:

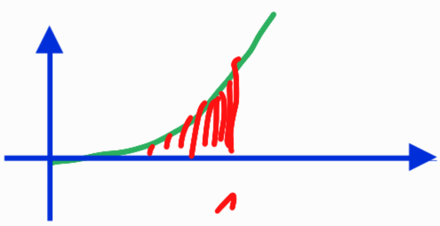
1) $$\int_0^1 x\,dx = \frac{1}{2}$$



2) $$\int_0^{2\pi} \sin x\,dx = 0$$



3) $$\int_0^1 x^2\,dx = ?$$



Hauptsatz der Diff. und Integralrechnung (HDI)

Def.: $\boxed{F \text{ ist Stammfunktion zu } f \;:\Longleftrightarrow\; F' = f}$

beachte: 1) mit F auch

$\tilde{F}(x) := F(x) + c$

Stammfkt. zu f

2) F und $\tilde{F}$ Stammfkt. zu f

$\rightarrow \quad (F - \tilde{F})' = 0$

$\rightarrow \quad F(x) - \tilde{F}(x) = c$

$\rightsquigarrow \quad F(x) = \tilde{F}(x) + c$

HDI

1)
$$F_{x_0}(x) := \int_{x_0}^{x} f(u)\,du$$

ist Stammfkt. zu f

2) Ist F Stammfkt. zu f,

dann

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b}$$

zu 1):

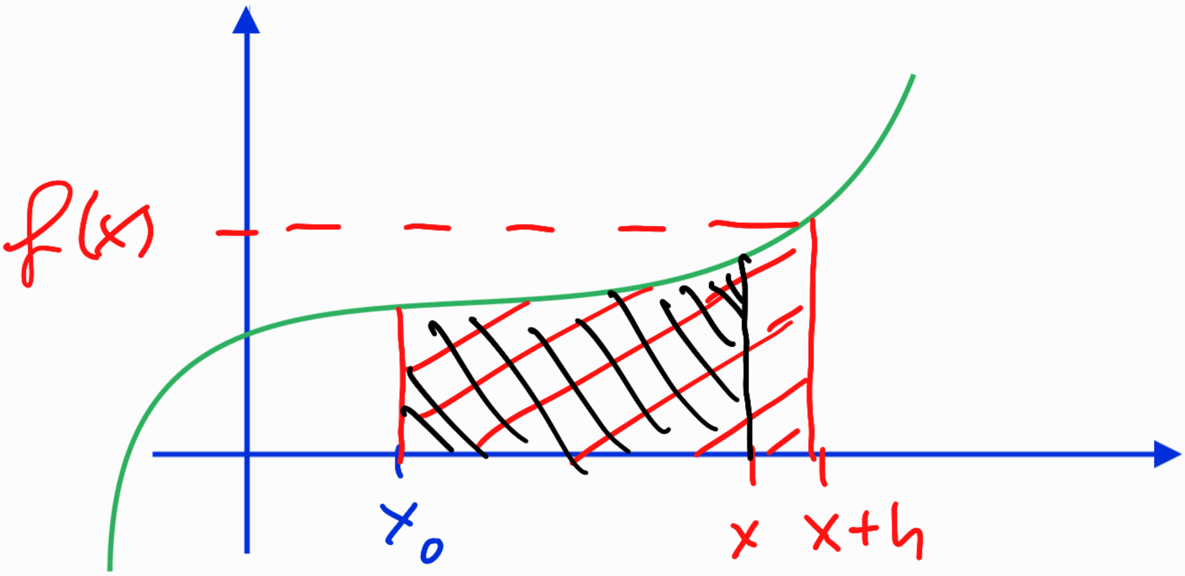
$$F_{x_0}(x) := \int_{x_0}^{x} f(u)\,du$$

z. z.: $F_{x_0}'(x) = f(x)$

$$F_{x_0}'(x) = \frac{1}{\cancel{h}}\Big(\underbrace{F_{x_0}(x+h) - F_{x_0}(x)}_{f(x)\cdot \cancel{h}}\Big) = f(x)$$

zu 2): F sei Stammfkt. zu f

z. z.: $\int_a^b f(x) = F(b) - F(a)$!



F und F_{x_0} unterscheiden sich nur in einer Konstanten!

mit $x_0 = a$ also

$$F(x) = F_a(x) + c$$

$$\Rightarrow F(b) - F(a) = F_a(b) + \cancel{c} - \underbrace{F_a(a)}_{=0} - \cancel{c}$$

$$= F_a(b) \overset{1)}{=} \int_a^b f(x)\,dx$$

✓

HDI:

$$F \underset{\text{integrieren}}{\overset{\text{ableiten}}{\rightleftarrows}} f$$

Integral ≙ „inverse Ableitung"

Integral = „Aufleitung"

<u>Hilfsmittel</u>:

f	$h'(x)$	x^{α}, $\alpha \neq -1$	$\frac{1}{x}$	e^{x}	$\sin x$
F	$h(x)$	$\frac{x^{\alpha+1}}{\alpha+1}$	$\ln x$	e^{x}	$-\cos x$

$$\boxed{\begin{aligned} f(x) &= h'(x)/h(x) \\ \rightarrow F(x) &= \ln h(x) \end{aligned}} \quad \xleftarrow{\frac{d}{dx}}$$

<u>Beispiele</u>:

1) $\displaystyle\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = 1/3$

2) $\displaystyle\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = 2$

3) $\displaystyle\int_a^b \sin(hx)\,dx = \frac{-\cos(hx)}{h}\Big|_a^b$

- $$\int_a^b \frac{1}{x}\,dx = \ln(b) - \ln(a)$$

- $$\int_a^b \tan x\,dx = \int_a^b \frac{\sin x}{\cos x}\,dx$$

$$= -\int_a^b \frac{(\cos x)'}{\cos x}\,dx$$

$$= -\ln(\cos x)\Big|_a^b \quad \checkmark$$

Partielle Integration:

$$\boxed{\int_a^b f'(x)\, g(x)\, dx = f(x)\, g(x)\Big|_a^b - \int_a^b f(x)\, g'(x)\, dx}$$

Lemma:

$$\int_a^b f' g \, dx \overset{!}{=} \int_a^b \{ (fg)' - f g' \} \, dx$$

$$= fg \Big|_a^b - \int_a^b f g' \, dx \quad !$$

Bsp:

$$\int_a^b e^x \cdot x \, dx = e^x x \Big|_a^b - \int_a^b e^x \, dx$$

$$= \left(e^x (x-1) \right) \Big|_a^b .$$

P.I.

$$\int_a^b 1 \cdot \ln x \, dx = x \ln x \Big|_a^b - \int_a^b \cancel{x} \frac{1}{\cancel{x}} \, dx$$

$f'(x)$ $\quad$ $g(x)$

$f(x) = x$

$$= \left(x \ln x - x \right) \Big|_a^b .$$