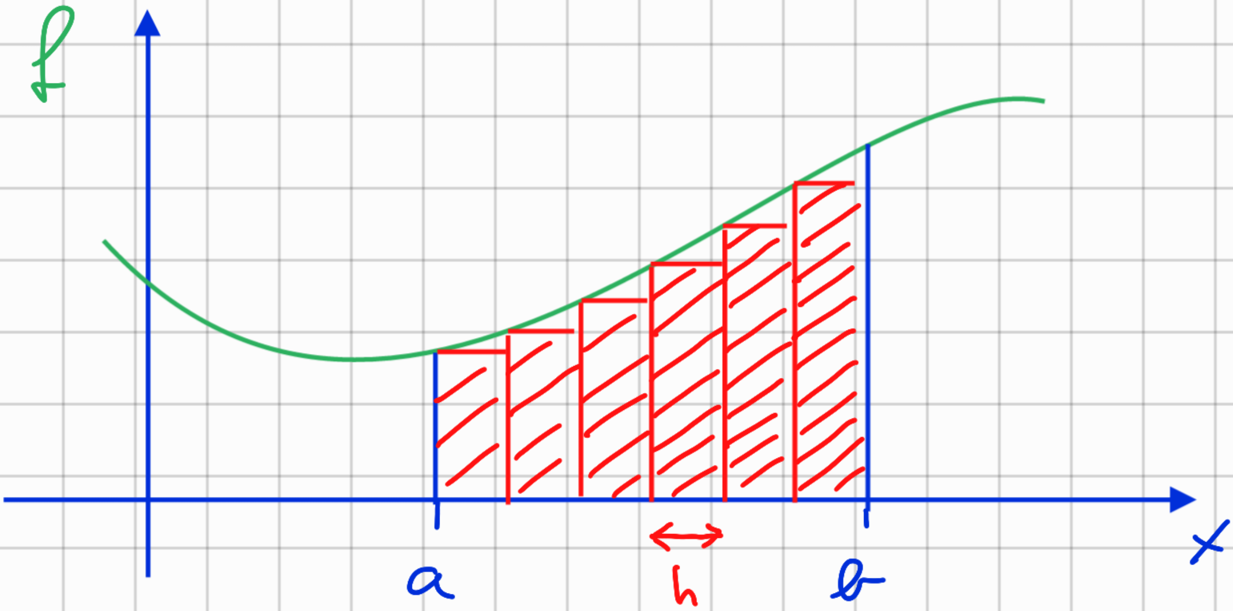
letzte Vrlsg.: <u>Integral</u>

- $$\int_a^b f(x)\,dx = A = \sum_{l=0}^{(b-a)/h} f(a+lh)\,h$$



- F <u>Stammfunktion</u> zu f

$$:\Longleftrightarrow \quad F' = f$$

- HDI:

1)
$$F_{x_0}(x) := \int_{x_0}^{x} f(x)\,dx$$

ist Stammfkt. zu f

2) Ist F Stammfkt. zu f, dann

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

- $$F \underset{\text{integrieren}}{\overset{\text{ableiten}}{\rightleftarrows}} f$$

- **partielle Integration:**

$$\int_a^b f'(x)\, g(x)\, dx = f(x) g(x) \Big|_a^b - \int_a^b f(x)\, g'(x)\, dx$$

Substitution

$$\int_a^b f(x)\, dx \overset{!}{=} \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(y))\, g'(y)\, dy$$

Subst.: $x = g(y)$

Idee:

$$\int_a^b f(x)\, dx = F(b) - F(a) =$$

$$= (F \circ g)(g^{-1}(b)) - (F \circ g)(g^{-1}(a))$$

$$= \int_{g^{-1}(a)}^{g^{-1}(b)} (F \circ g)'(y)\, dy \quad \checkmark$$

$$\overset{?}{=} f(g(y))\, g'(y) \quad \lrcorner$$

Bsp.:

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx = \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin^2 y}} \cos y \, dy$$

$$x = \sin y \qquad \frac{1}{\cos y}$$

$$= \int_0^{\pi/2} 1 \, dy = \frac{\pi}{2}$$

Uneigentliche Integrale anhand von Beispielen:

1) $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \underbrace{\int_1^b \frac{1}{x^2}\,dx}_{-\frac{1}{x}\Big|_1^b}$$

$$= \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1 \quad ✓$$

2) $$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty}\left(\ln b - \underset{\substack{\shortparallel\\0}}{\ln 1}\right) = \infty\,!$$

3)

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon \searrow 0} \underbrace{\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx}_{2\sqrt{x}\,\big|_\varepsilon^1}$$

$$= \lim_{\varepsilon \searrow 0} \left(2 - 2\sqrt{\varepsilon}\right) = 2$$

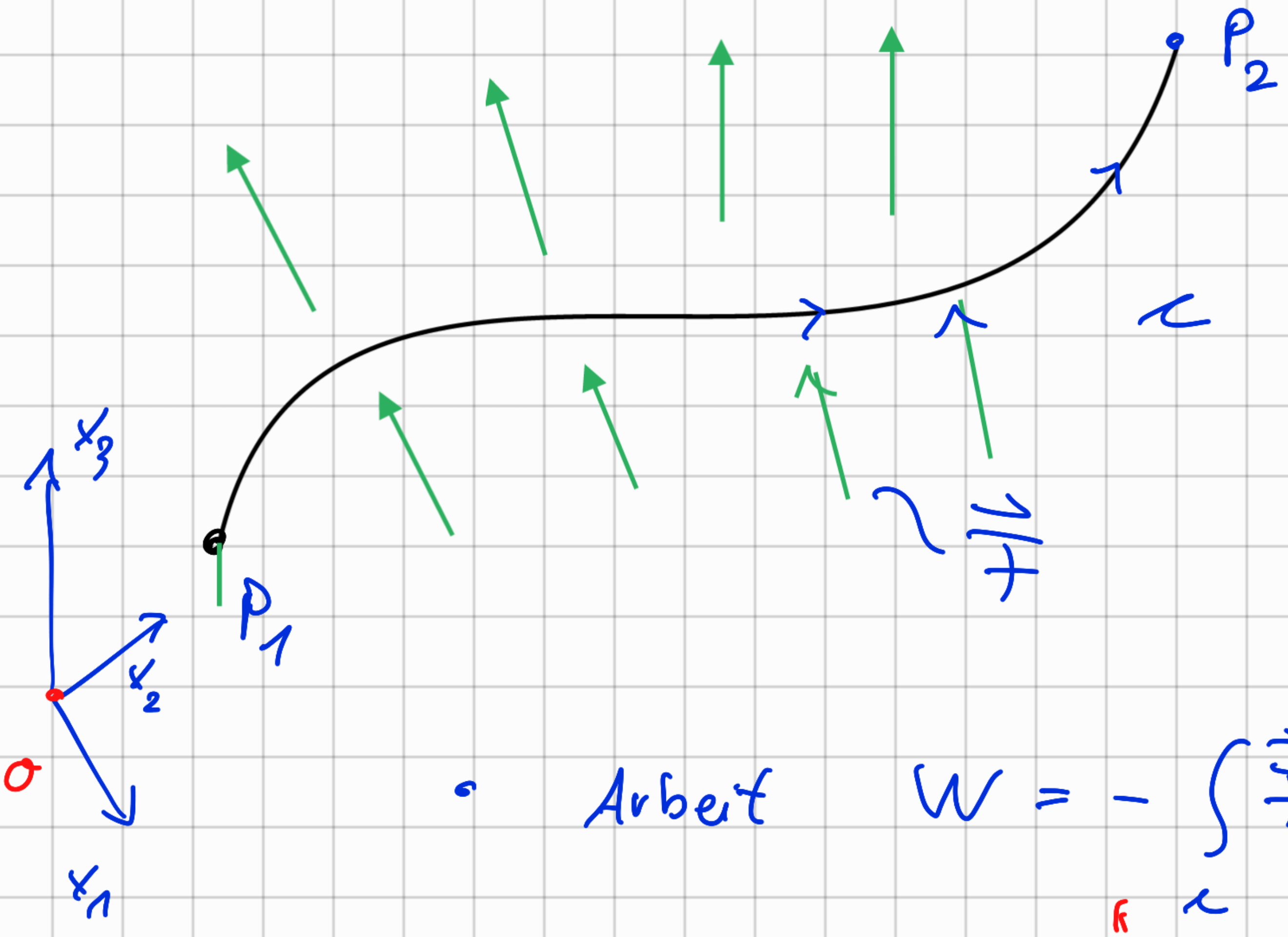
4)

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{\varepsilon \searrow 0} \underbrace{\int_\varepsilon^1 \frac{1}{x^2}\,dx}_{-\frac{1}{x}\big|_\varepsilon^1 = \frac{1}{\varepsilon} - 1} = \infty \,!$$

$$= \lim_{\varepsilon \searrow 0} \left(\frac{1}{\varepsilon} - 1\right) = \infty \,!$$

Wege und Wegintegrale

Motivation: Teilchen werde im Kraftfeld $\vec{F}(\vec{r})$ längs eines Wegs c von P_1 nach P_2 bewegt:



- Arbeit $W = -\int_c \vec{F} \, d\vec{l} = ?$

- Länge des Wegs:

$$|c| = \int_c dl = ?$$

- Weg ?

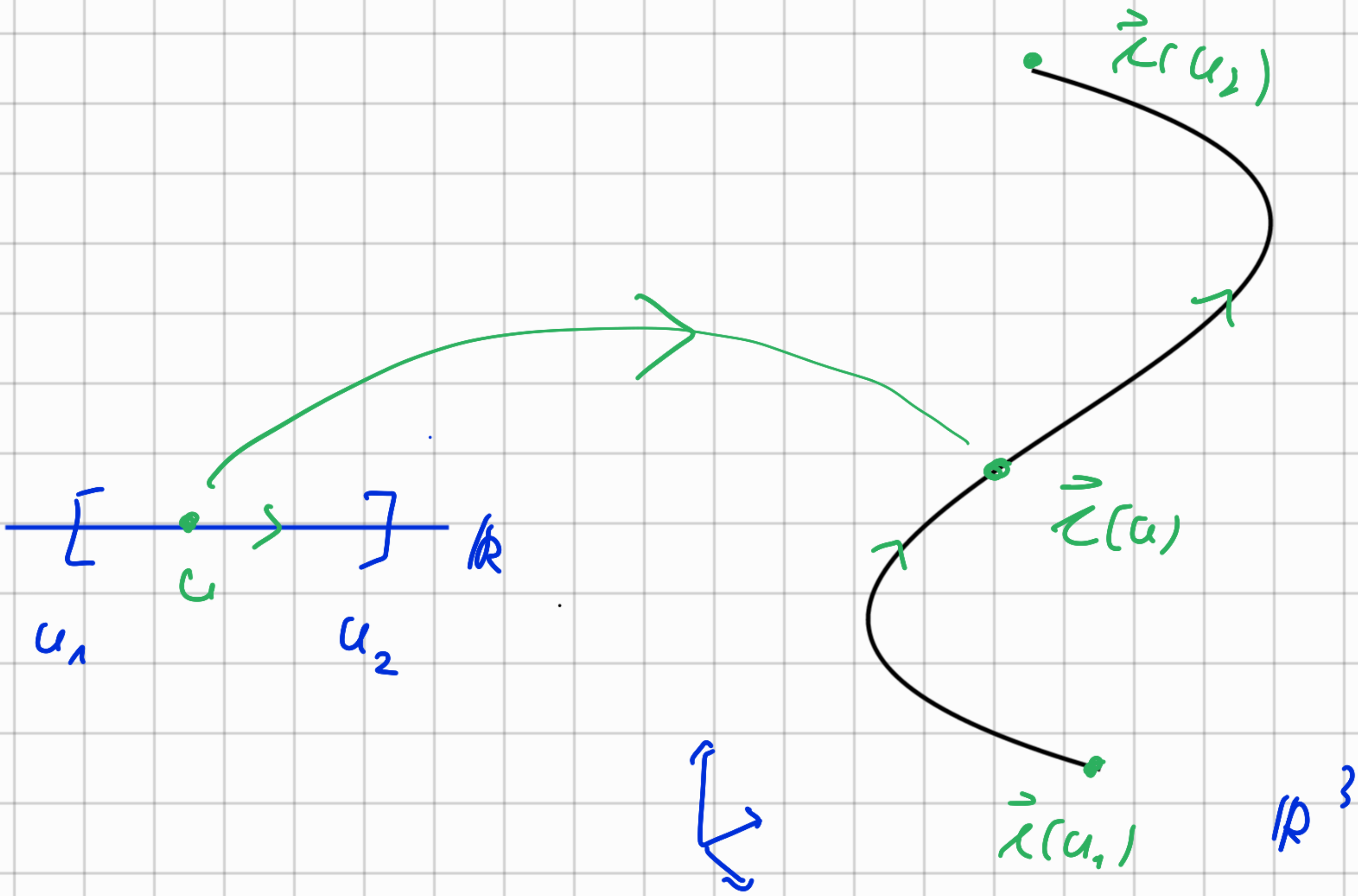
- Kraftfeld, Vektorfeld ?

<u>Weg c im $\mathbb{R}^3$</u> :

Parametrisierung

$\equiv$ Abb. $c : [u_1, u_2] \longrightarrow \mathbb{R}^3$

$u \longmapsto \vec{c}(u)$



Bsp.:

Schraubenweg:

n Windungen

Radius R

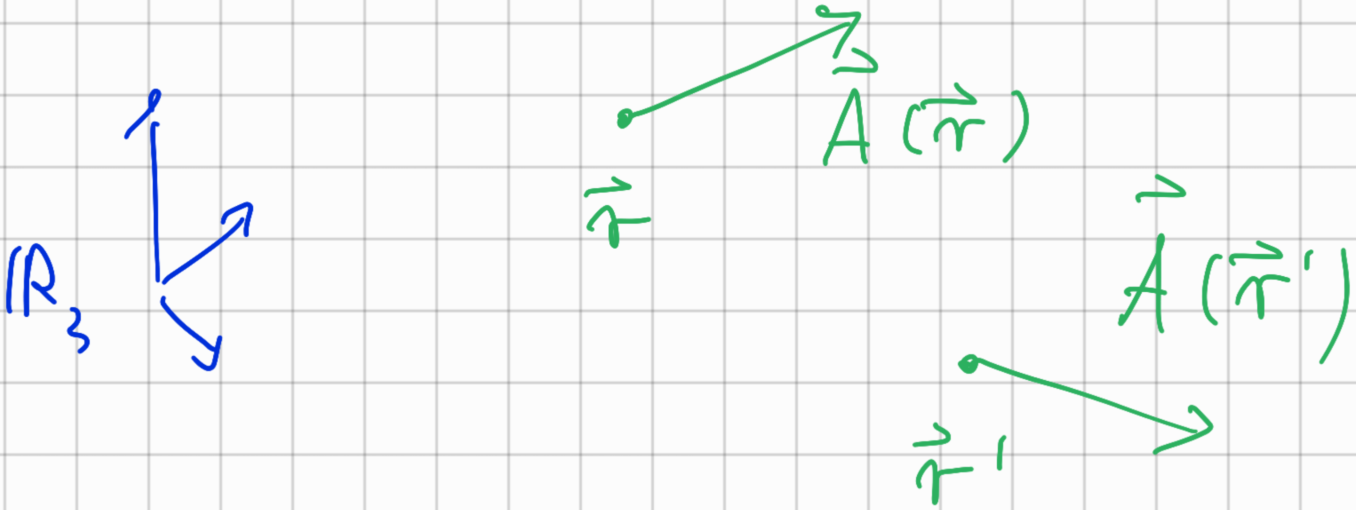
Höhe H

$$c : [0, 2\pi n] \to \mathbb{R}^3$$

$$u \mapsto \begin{pmatrix} R\cos u \\ R\sin u \\ \frac{H}{2\pi n}\, u \end{pmatrix}$$

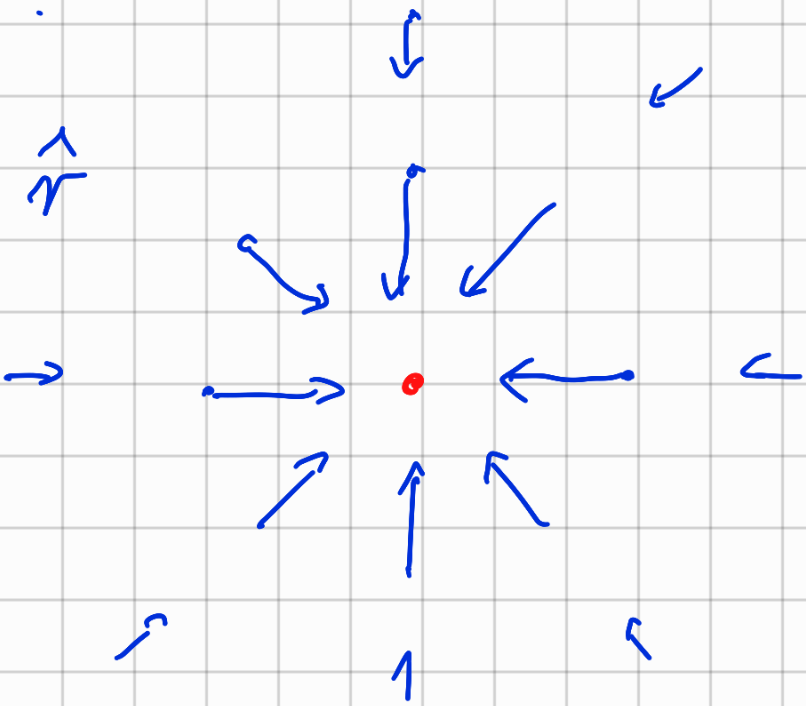
Vektorfeld im $\mathbb{R}^3$:

$$\vec{A} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\vec{r} \longmapsto \vec{A}(\vec{r})$$



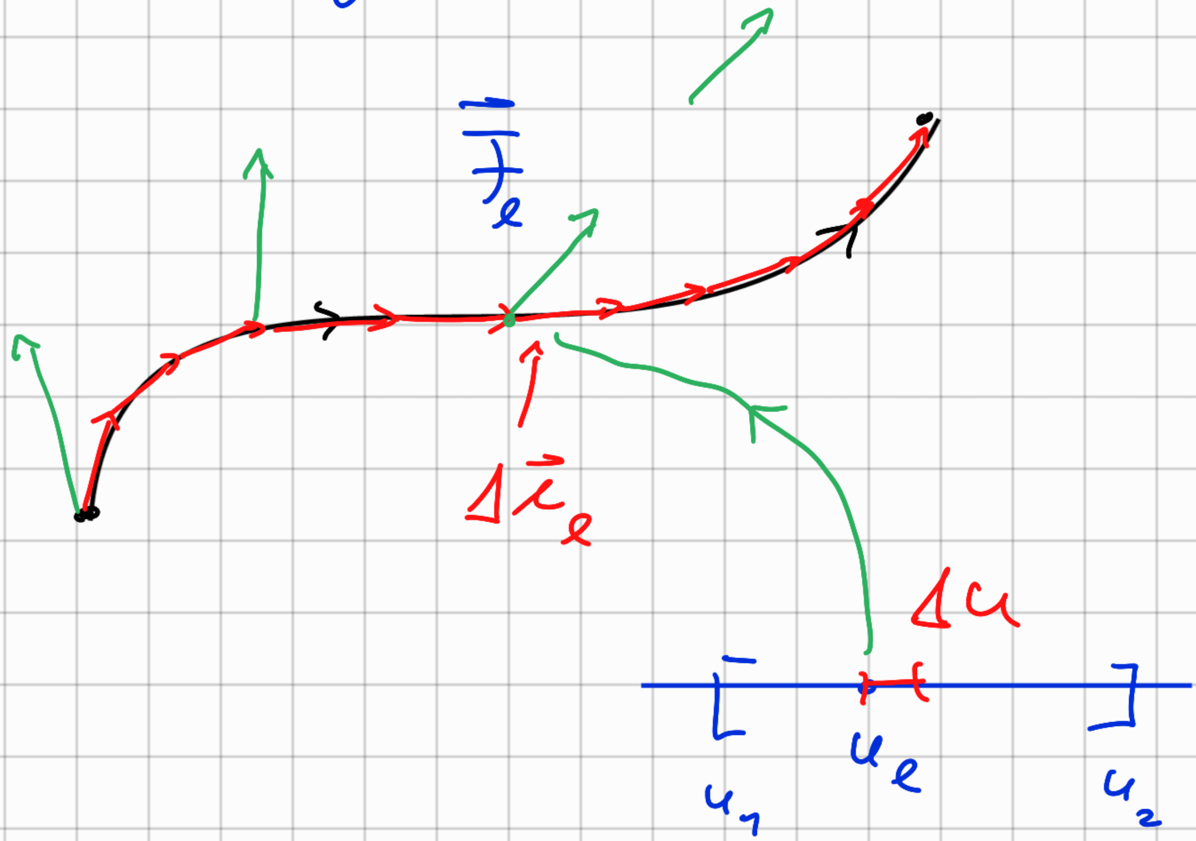
Bsp.: Gravitationsfeld einer Punktmasse M in O:

$$\vec{g}(\vec{r}) := -\frac{GM}{r^2}\hat{r}$$



Wegintegral über Kraftfeld $\vec{F}$

längs Weg c:



$$\int_c \vec{F}\, d\vec{l} \overset{!}{=} \sum_{\ell=0}^{n} \langle \vec{F}_\ell , \Delta\vec{r}_\ell \rangle$$

- $\Delta\vec{r}_\ell = \frac{d\vec{r}(u_\ell)}{du} \Delta u$
- $\vec{F}_\ell = \vec{F}(\vec{r}(u_\ell))$

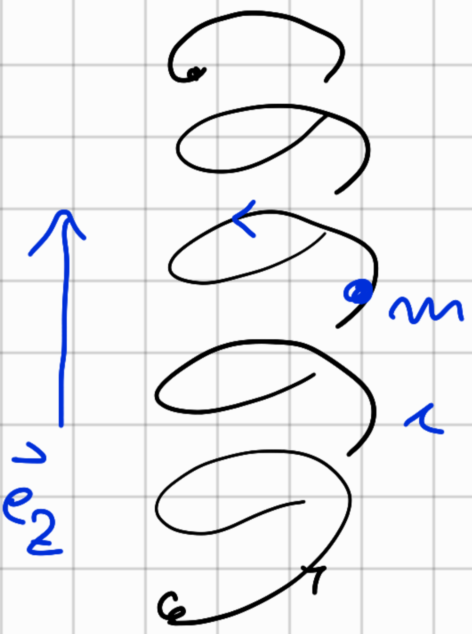
$$\int_{\mathcal{C}} \vec{F}\, d\vec{l} = \sum_{l=0}^{n} \left\langle \vec{F}(\vec{r}(u_l)), \frac{d\vec{r}}{du}(u_l) \right\rangle \Delta u$$

$\Delta u \to 0$:

Def :

$$\int_{\mathcal{C}} \vec{A}\, d\vec{l} := \int_{u_1}^{u_2} \left\langle \vec{A}(\vec{r}(u)), \frac{d\vec{r}}{du}(u) \right\rangle du$$

Bsp.:



Gravitationskraft:

$$\vec F(\vec r) = m\vec g = -mg\,\vec e_z$$

$$c : [0, 2\pi n] \to \mathbb{R}^3$$

$$u \longmapsto \begin{pmatrix} R\cos u \\ R\sin u \\ \frac{H}{2\pi n}\, u \end{pmatrix}$$

$$W = -\int_c \vec F \, d\vec l$$

$$\begin{pmatrix} -R\sin u \\ R\cos u \\ H/2\pi n \end{pmatrix}$$

$$\overset{\text{Def.}}{=} -\int_0^{2\pi n} \Big\langle \underbrace{\vec F(\vec c(u))}_{-mg\,\vec e_z},\ \overbrace{\frac{d\vec c}{du}(u)} \Big\rangle \, du$$

$$= + mg \int_0^{2\pi n} \frac{H}{2\pi n}\, du$$

$$\rightarrow \quad W = mgH \,.$$