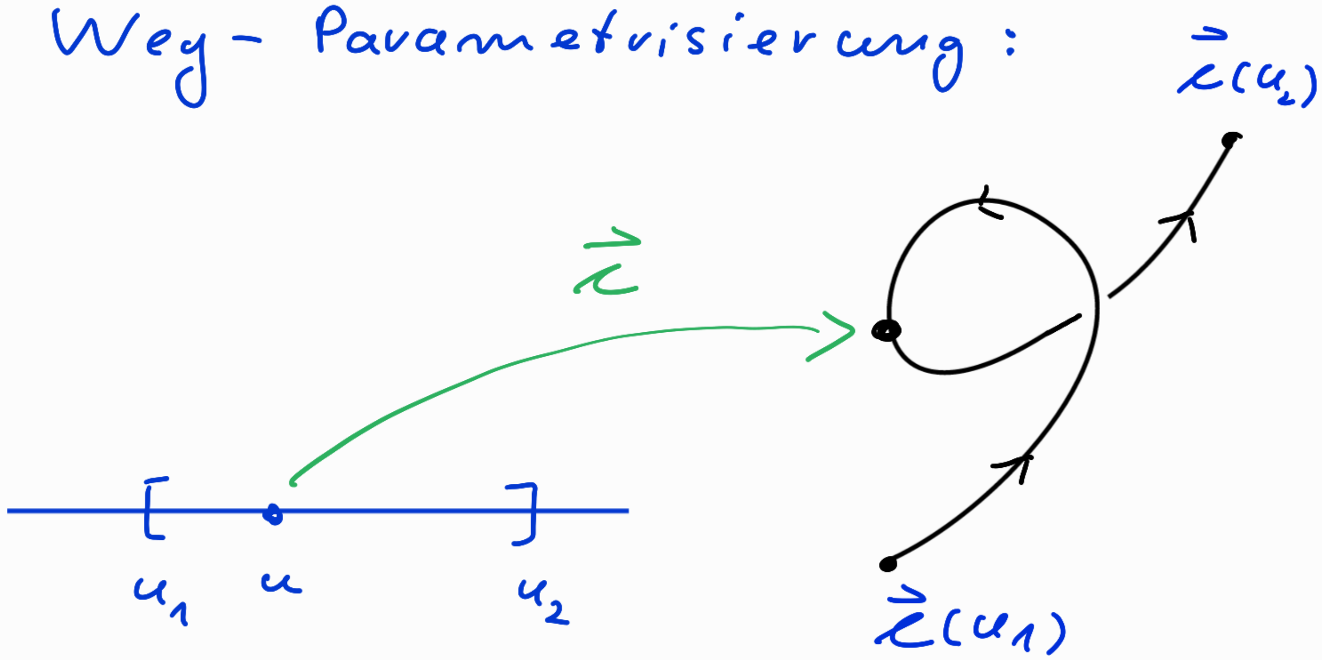
letzte Vrlsg.:

- Weg-Parametrisierung:

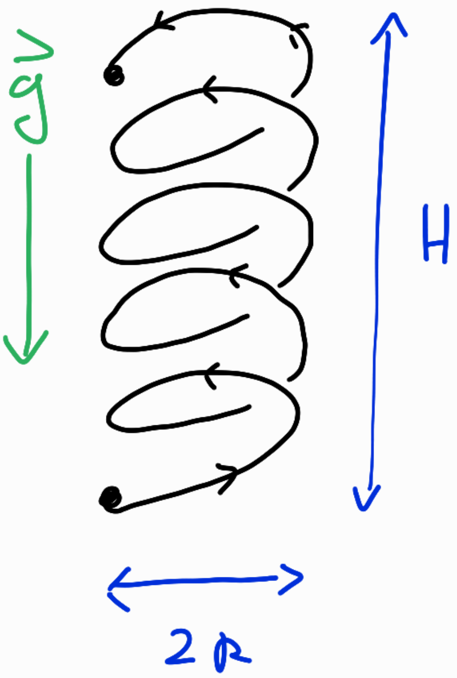


$$\vec{r} : [u_1, u_2] \longrightarrow \mathbb{R}^3$$
$$u \longmapsto \vec{r}(u)$$

- Wegintegral eines Vektorfelds $\vec{A}$ längs Weg r:

$$\boxed{\int_r \vec{A}\, d\vec{l} := \int_{u_1}^{u_2} \left\langle \vec{A}(\vec{r}(u)), \frac{d\vec{r}}{du}(u) \right\rangle du}$$

Bsp.:



$$\vec r : [0, 2\pi n] \to \mathbb{R}^3$$

$$\varphi \longmapsto \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ \frac{H\varphi}{2\pi n} \end{pmatrix}$$

$$\vec g = -g\,\vec e_z$$

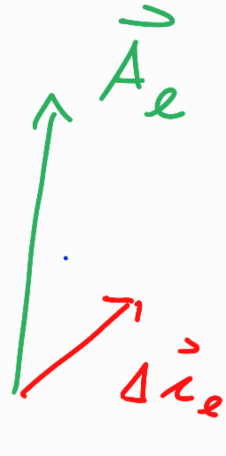
$$\leadsto \vec F = m\vec g$$

$$\to \quad W = -\int_r \vec F\, d\vec \ell$$

$$= -\int_0^{2\pi n} \left\langle m\vec g, \frac{d\vec r}{d\varphi}(\varphi) \right\rangle d\varphi$$

$$= \dots = mgH$$

$$\hat= \lim_{|\Delta \vec r| \to 0} \sum_{\ell=1}^{n} \langle \vec A_\ell, \Delta \vec r_\ell \rangle$$



Integral einer Fkt.

$$f: \mathbb{R}^3 \to \mathbb{R}, \quad \vec{r} \mapsto f(\vec{r})$$

("Skalarfeld") längs Weg c:

$$\boxed{\int_c f \, dl := \int_{u_1}^{u_2} f(\vec{c}(u)) \left| \frac{d\vec{c}(u)}{du} \right| du}$$

$f = 1$: $\to$ Länge des Wegs c:

$$\boxed{\underline{\underline{L(c)}} = \int_c 1 \, dl = \int_{u_1}^{u_2} \left| \frac{d\vec{c}(u)}{du} \right| du}$$

Bsp.: Länge des Schraubenwegs:

$$\gamma : [0, 2\pi n] \longrightarrow \mathbb{R}^3$$

$$\varphi \longmapsto \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ \frac{H}{2\pi n}\varphi \end{pmatrix}$$

$$\vec{\gamma}\,'(\varphi) = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ H/2\pi n \end{pmatrix}$$

$$\to |\vec{\gamma}(\varphi)| = \left(R^2 + \frac{H^2}{(2\pi n)^2} \right)^{1/2}$$

$$\to L(\gamma) = \int_0^{2\pi n} |\vec{\gamma}\,'(\varphi)|\, d\varphi$$

$$= \int_0^{2\pi n} \sqrt{R^2 + \left(\frac{H}{2\pi n}\right)^2}\, d\varphi$$

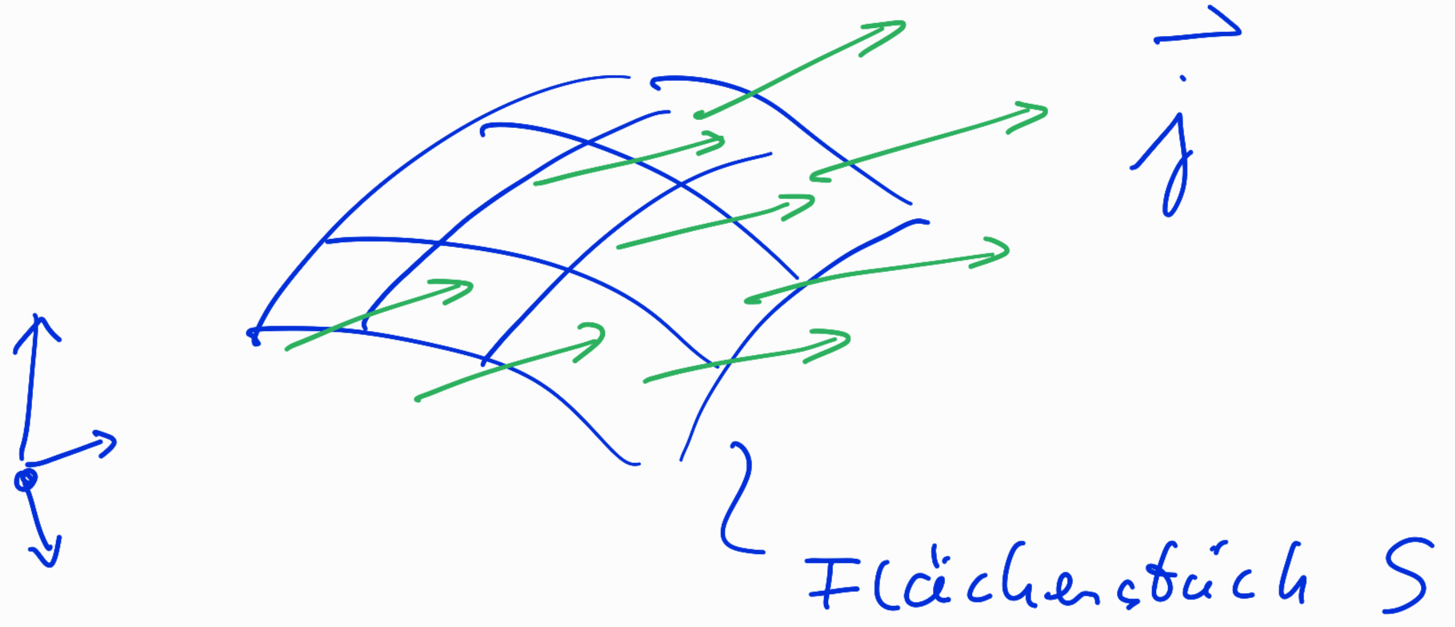
d.h. $\boxed{L(\gamma) = \sqrt{(2\pi R n)^2 + H^2}}$

$H = 0$: $L(\gamma) = n\, 2\pi R$ ✓

$R = 0$: $L(\gamma) = H$ ✓

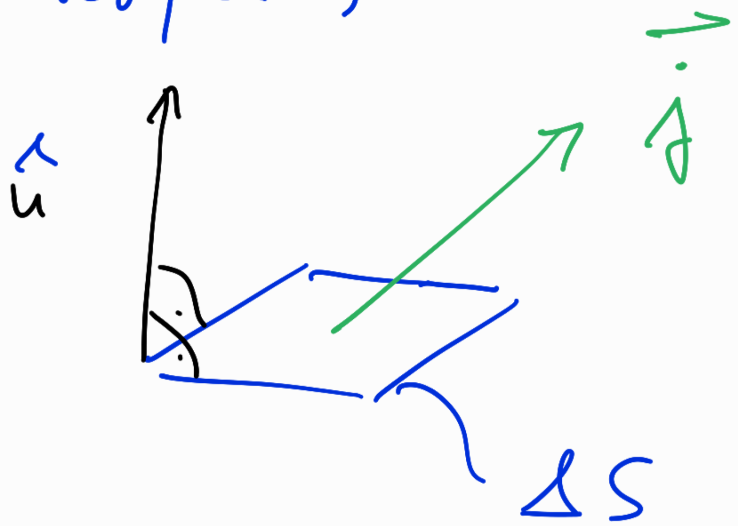
Flächen und Flächenintegrale

Motivation:



Massenstromdichte $\vec{j}(\vec{r})$

(Vektorfeld)



Massenstrom durch ΔS:

$$\Delta I = \langle \vec{j}, \hat{n} \rangle \Delta S$$

$$\vec{\Delta S} := \hat{n}\, \Delta S$$

$$\rightarrow \quad \Delta I = \langle \vec{j}, \vec{\Delta S} \rangle$$

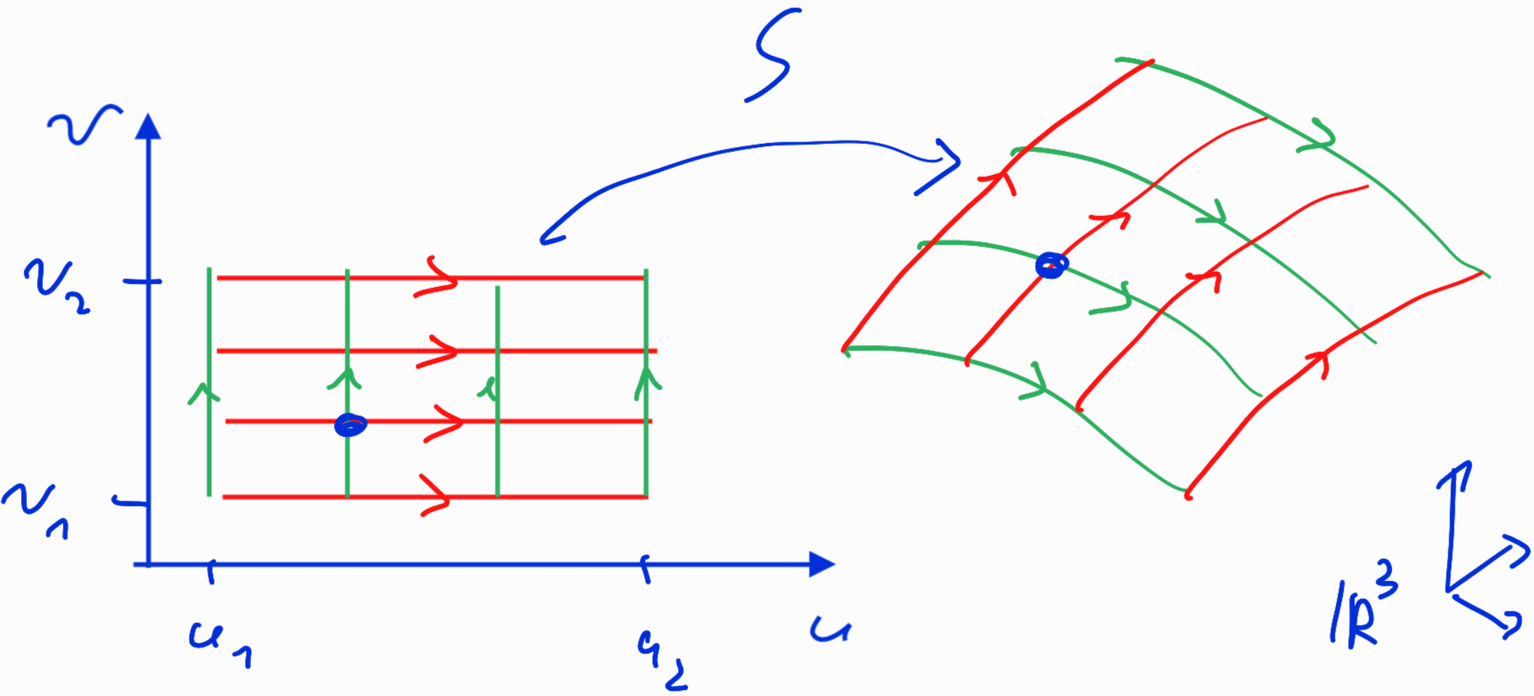
2) Gesamtstrom durch S:

$$I = \int_S \vec{j}\, \vec{df} := \sum_{e=1}^{4} \Delta I_e$$

Parametrisierung einer Fläche:

Abb: $S : [u_1, u_2] \times [v_1, v_2] \to \mathbb{R}^3$

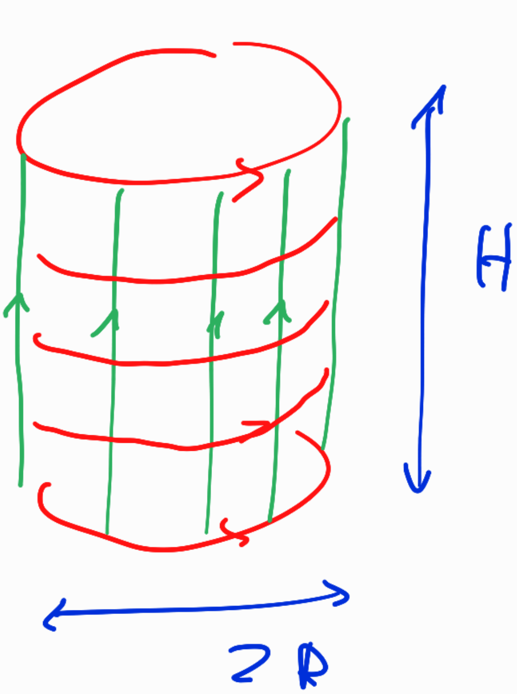
$(u, v) \longmapsto \vec{S}(u, v)$



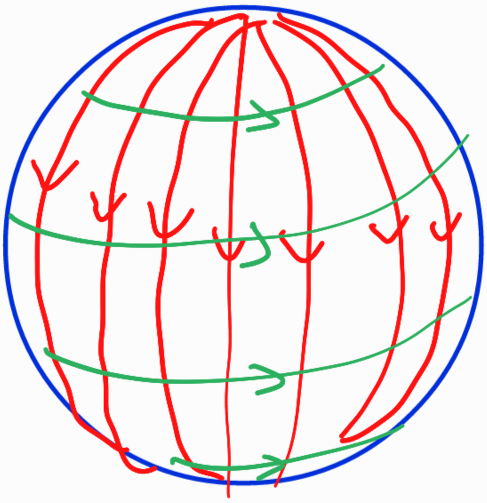
<u>Beispiele</u>: Zylindermantel

$Z : [0, 2\pi] \times [0, H] \to \mathbb{R}^3$

$$(\varphi, z) \longmapsto \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix}$$



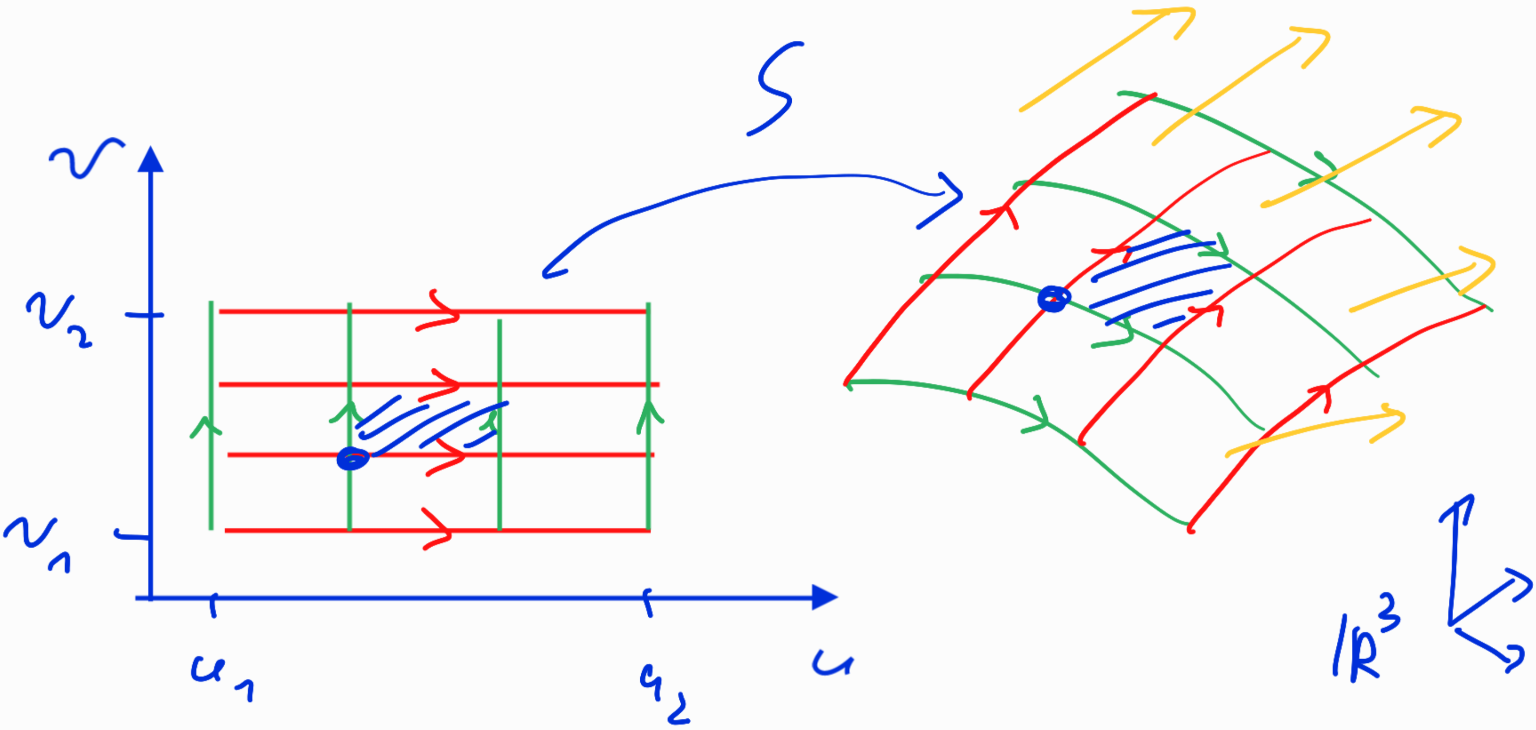
Sphäre: $S : [0,\pi] \times [0,2\pi] \to \mathbb{R}^3$

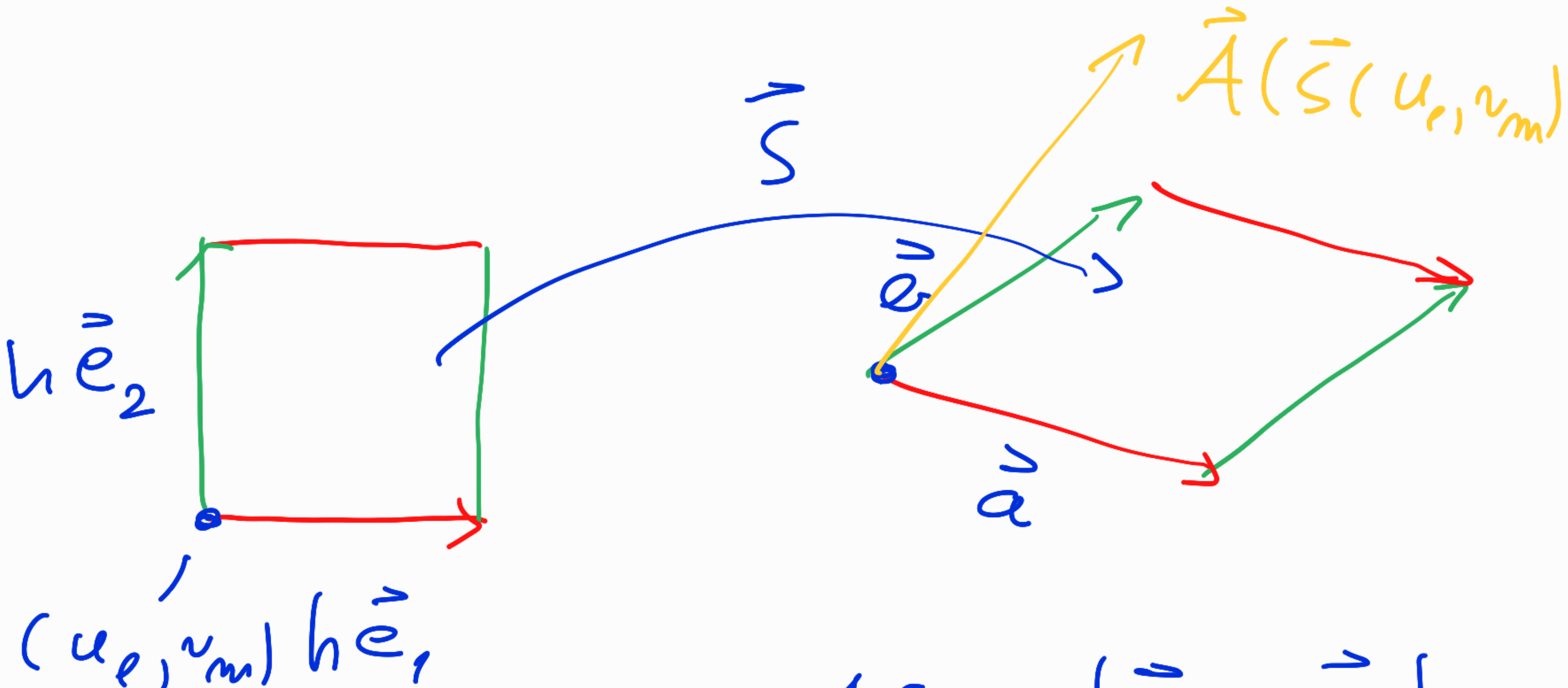


$$(\vartheta,\varphi) \mapsto R\begin{pmatrix}\cos\varphi\,\sin\vartheta \\ \sin\varphi\,\sin\vartheta \\ \cos\vartheta\end{pmatrix}$$

Flächenintegral

$\vec{A}$





$(h \to 0)$

$$\Delta S = |\vec{a} \times \vec{b}|$$

$$\vec{\Delta S} = \vec{a} \times \vec{b}$$

$$\vec{a} = \frac{\partial \vec{S}}{\partial u}(u_l, v_m)\, h$$

$$\vec{b} = \frac{\partial \vec{S}}{\partial v}(u_l, v_m)\, h$$

$$\to \quad \vec{\Delta S} = \frac{\partial \vec{S}}{\partial u} \times \frac{\partial \vec{S}}{\partial v} \cdot h^2$$

$(*)$

$(\vec{S} = \vec{S}(u_l, v_m))$

$$\Delta I_{\ell m} = \langle \vec{A}(\vec{S}), \Delta \vec{S} \rangle$$

$$\overset{*}{=} \langle \vec{A}(\vec{S}), \frac{\partial \vec{S}}{\partial u} \times \frac{\partial \vec{S}}{\partial v} \rangle h^2$$

$$\sum_{\ell, m} \Delta I_{\ell m} \qquad h \to 0 :$$

$$\boxed{\int_S \vec{A}\, d\vec{f} := \int_{u_1}^{u_2} \int_{v_1}^{v_2} \langle \vec{A}(\vec{S}), \frac{\partial \vec{S}}{\partial u} \times \frac{\partial \vec{S}}{\partial v} \rangle \, du\, dv}$$

$$\vec{S} = \vec{S}(u, v)$$

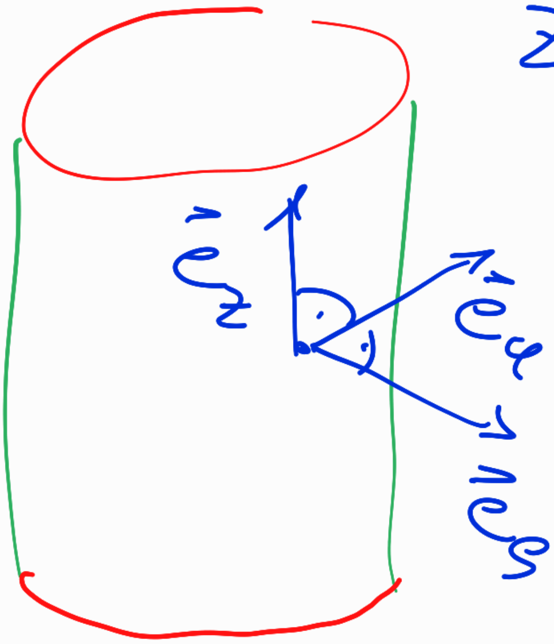
Flächenintegral eines Skalarfelds $f: \mathbb{R}^3 \to \mathbb{R}$:

$$\boxed{\int_S f \, df := \int_{u_1}^{u_2} \int_{v_1}^{v_2} f(\vec{S}) \left| \frac{\partial \vec{S}}{\partial u} \times \frac{\partial \vec{S}}{\partial v} \right| du \, dv}$$

$f = 1$: <u>Flächeninhalt von S</u>:

$$A(S) = \int_S 1 \, df = \int_{u_1}^{u_2} \int_{v_1}^{v_2} \left| \frac{\partial \vec{S}}{\partial u} \times \frac{\partial \vec{S}}{\partial v} \right| du \, dv$$

Beispiele : Zylindermantel :



$$Z : [0, 2\pi] \times [0, H] \to \mathbb{R}^3$$

$$(\varphi, z) \mapsto \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix}$$

$$\frac{\partial \vec{Z}}{\partial \varphi} = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix} = R\,\vec{e}_\varphi$$

$$\frac{\partial \vec{Z}}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \vec{e}_z$$

$$\frac{\partial \vec{Z}}{\partial \varphi} \times \frac{\partial \vec{Z}}{\partial z} = R\,\vec{e}_\varphi \times \vec{e}_z = R\,\vec{e}_\rho$$

$\rightarrow$

$$\int_Z \vec{A}\, d\vec{f} = \int_0^{2\pi}\int_0^H \underbrace{\overbrace{\langle \vec{A}(\vec{z}(\varphi,z)), \vec{e}_\varsigma \rangle}^{A_\varsigma(\varphi,z)}}_{R\, d\varphi\, dz}$$

$$\int_Z \vec{A}\, d\vec{f} = \int_0^{2\pi}\int_0^H A_\varsigma(\varphi,z)\, \underbrace{\underline{R}\, d\varphi\, dz}$$

„Flächenelement"

Flächeninhalt des Zylindermantels:

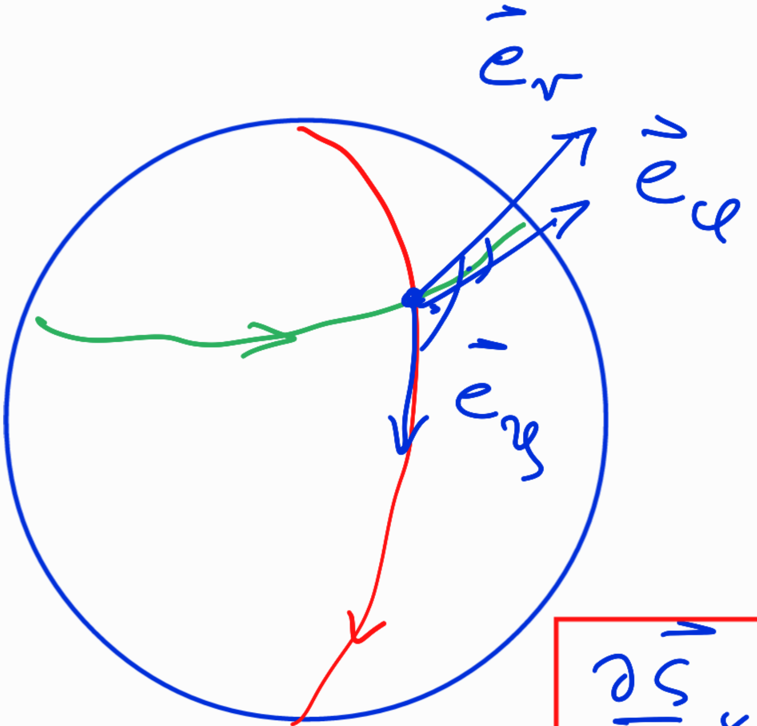
$$\int_Z 1\, df = \int_0^{2\pi}\int_0^H 1\ R\, d\varphi\, dz =$$

$$= \int_0^{2\pi} \underbrace{\left(\int_0^H R\, dz \right)}_{= HR} d\varphi = 2\pi R H \quad \checkmark$$

Sphäre:

$$S : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$$

$$(\vartheta, \varphi) \mapsto R \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix}$$



$$\frac{\partial \vec{S}}{\partial \vartheta} = R \vec{e}_\vartheta$$

$$\frac{\partial \vec{S}}{\partial \varphi} = R \sin\vartheta \, \vec{e}_\varphi$$

$$\boxed{\frac{\partial \vec{S}}{\partial \vartheta} \times \frac{\partial \vec{S}}{\partial \varphi} = R^2 \sin\vartheta \, \vec{e}_r}$$

$\rightsquigarrow$

$$\int_S \vec{A}\, d\vec{f} = \int_0^{\pi}\int_0^{2\pi} A_r(\vartheta,\varphi) \underbrace{R^2 \sin\vartheta \, d\vartheta \, d\varphi}_{\text{Flächenelement}}$$

Flächeninhalt der Sphäre:

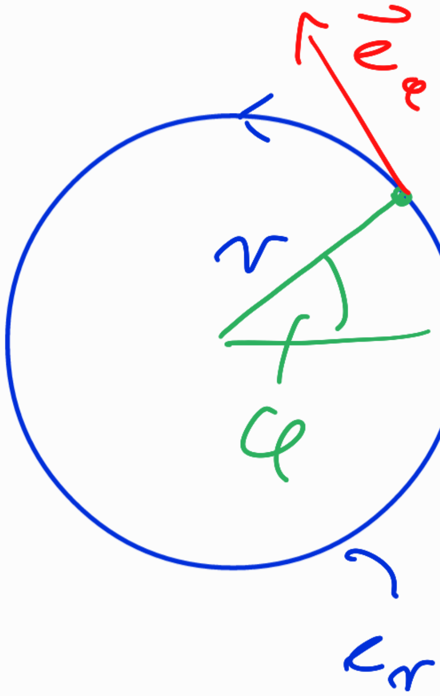
$$\int_S 1\, df = \int_0^{\pi}\int_0^{2\pi} R^2 \sin\vartheta \, d\vartheta \, d\varphi$$

$$= 2\pi R^2 \underbrace{\int_0^{\pi} \sin\vartheta \, d\vartheta}_{-\cos\vartheta \big|_0^{\pi}} =$$

$$= 2\pi R^2 \left(+1 + 1 \right) = \underline{\underline{4\pi R^2}} \; !$$

Nachtrag:

Standardparametrisierung des Kreiswegs:



$$c_r : [0, 2\pi] \to \mathbb{R}^2$$

$$\varphi \mapsto r \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$c_r' = r \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$$

$$= r\, \vec{e}_\varphi$$

$\Rightarrow$

$$\int_{C_r} \vec{A}\, d\vec{l} = \int_0^{2\pi} \underbrace{\overbrace{\langle \vec{A}(\vec{e}_r(\varphi)), \vec{e}_\varphi \rangle}^{A_\varphi(\varphi)}}_{r\, d\varphi}$$

$$\int_{C_r} \vec{A}\, d\vec{l} = \int_0^{2\pi} A_\varphi(\varphi)\, \underbrace{r\, d\varphi}_{\text{Linien-element}}$$