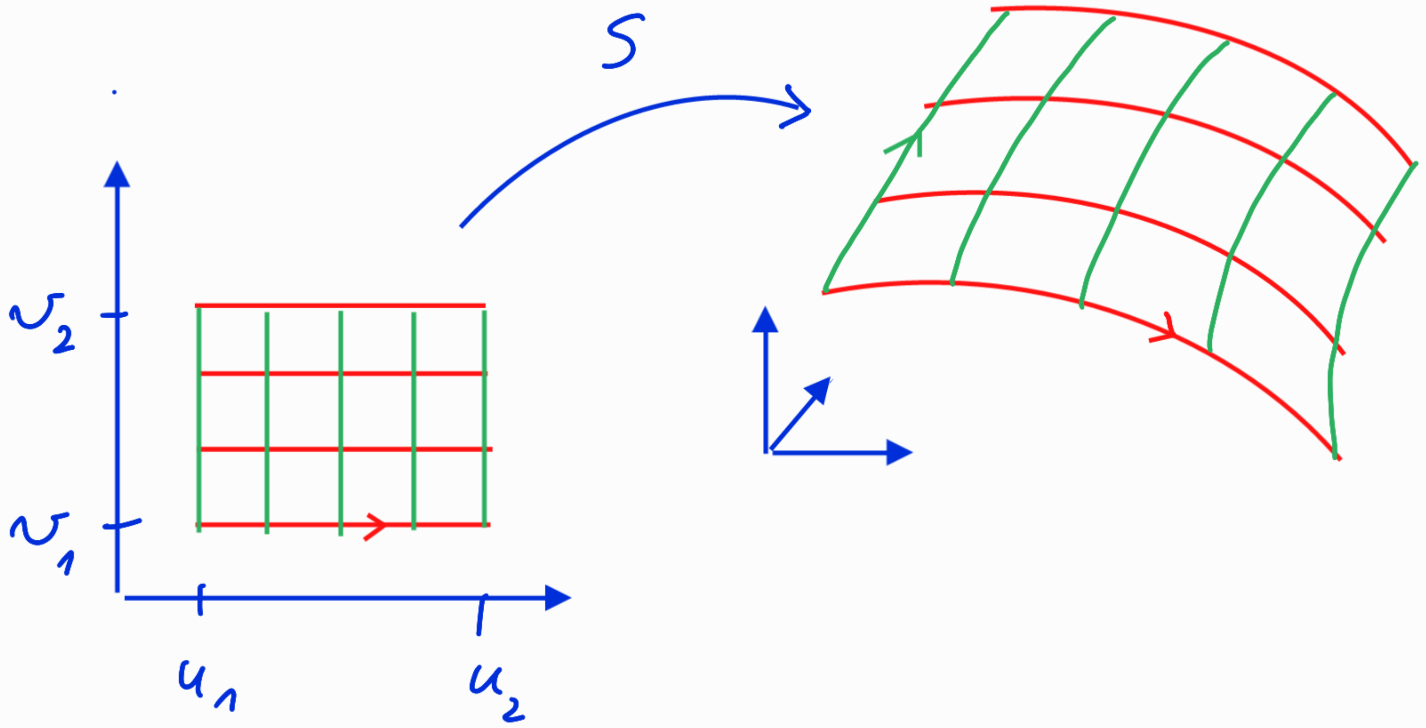
letzte Vrlsg.:

- Parametrisierung einer Fläche:



$$S : [u_1, u_2] \times [v_1, v_2] \longrightarrow \mathbb{R}^3$$
$$(u, v) \longmapsto \vec{S}(u, v)$$

- Flächenintegrale:

$$\int_S \vec{A}\, d\vec{f} := \int_{u_1}^{u_2} \int_{v_1}^{v_2} \left\langle \vec{A}(\vec{S}),\ \frac{\partial \vec{S}}{\partial u} \times \frac{\partial \vec{S}}{\partial v} \right\rangle dv\, du$$

$$\vec{S} = \vec{S}(u, v)$$

$$\int_S f\, df = \int_{u_1}^{u_2} \int_{v_1}^{v_2} f(\vec{s}) \left| \frac{\partial \vec{s}}{\partial u} \times \frac{\partial \vec{s}}{\partial v} \right| du\, dv$$

- Integral über Zylindermantel:

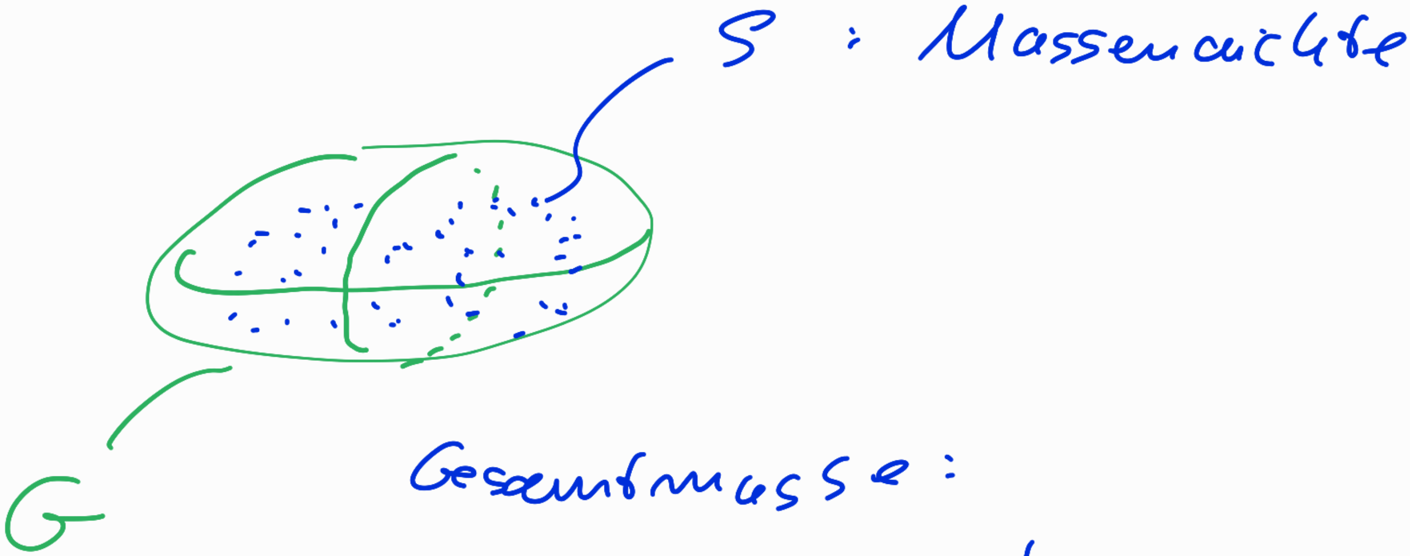
$$\int_{Z_{R,H}} \vec{A}\, d\vec{f} = \int_0^H \int_0^{2\pi} A_\varrho(\varphi, z)\, R\, d\varphi\, dz$$

- über Sphäre:

$$\int_{S_R} \vec{A}\, d\vec{f} = \int_0^{2\pi} \int_0^{\pi} A_r(\varphi, \vartheta)\, R^2 \sin\vartheta\, d\vartheta\, d\varphi$$

Volumengebiete und Volumenintegr.

Motivation:



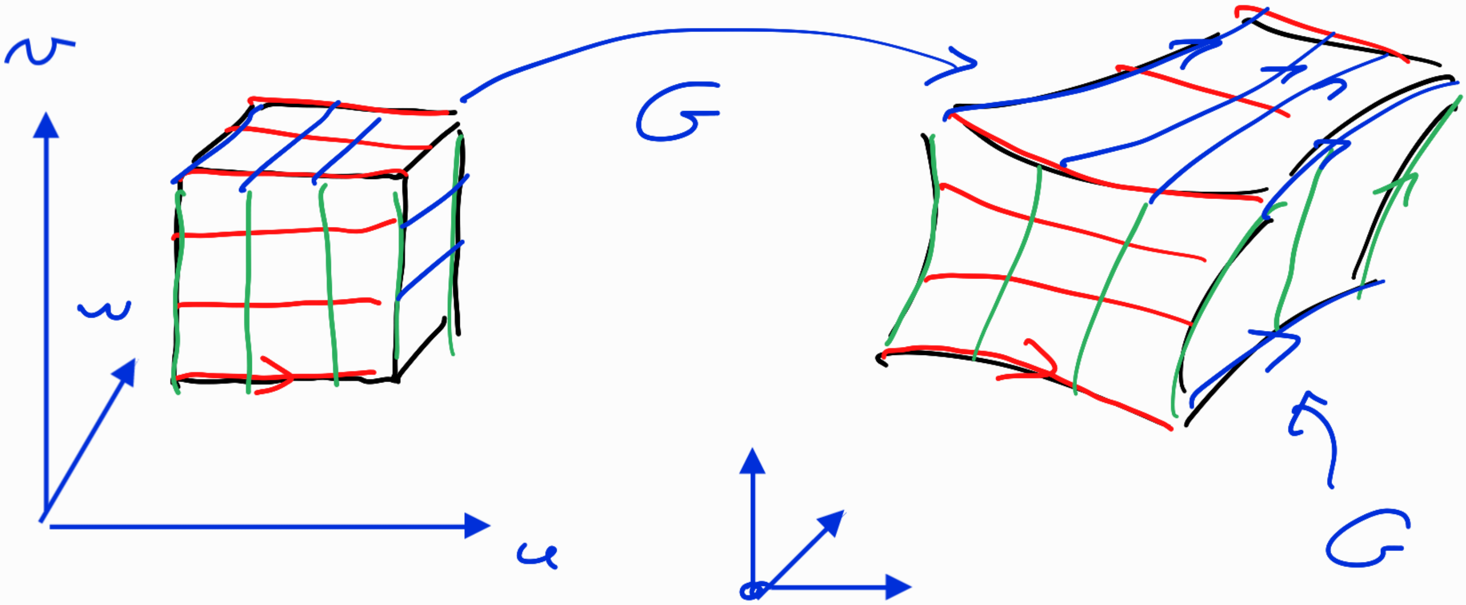
Gesamtmasse:

$$M_G = \int_G \varrho \, dV \overset{!}{=} \ldots$$

Parametrisierung eines Gebiets $G \subset \mathbb{R}^3$

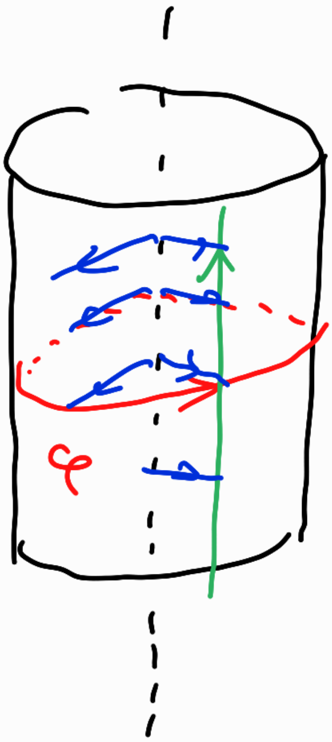
$$\vec{G} : [u_1, u_2] \times [v_1, v_2] \times [w_1, w_2] \to \mathbb{R}^3$$

$$(u, v, w) \longmapsto \vec{G}(u, v, w)$$



Beispiele: 1) Vollzylinder:

$$Z : [0, 2\pi] \times [0, H] \times [0, R] \to \mathbb{R}^3$$

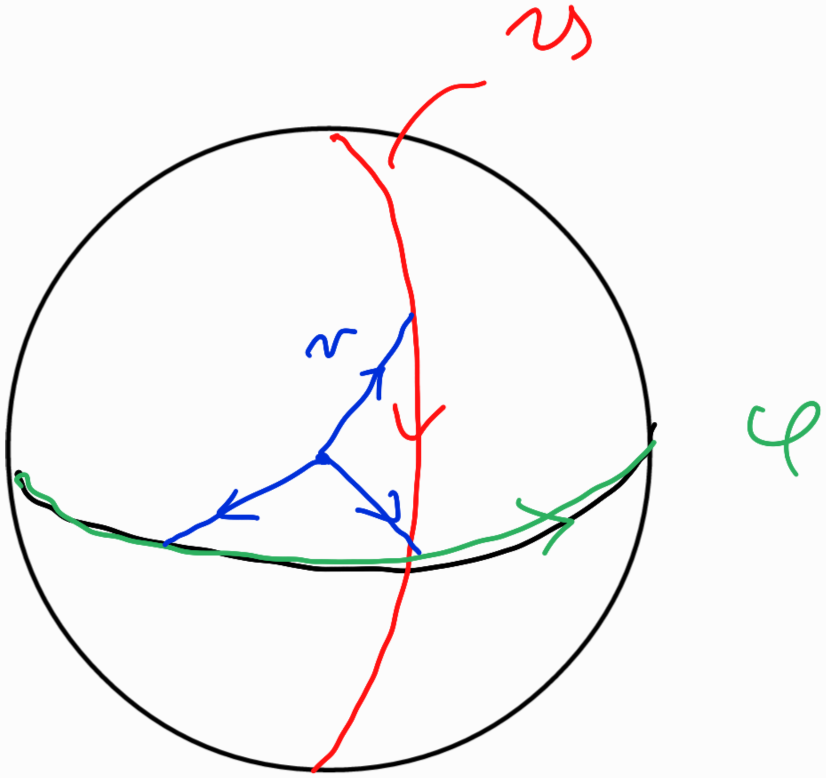


$$(\varphi, z, \rho) \mapsto \begin{pmatrix} \rho \cos\varphi \\ \rho \sin\varphi \\ z \end{pmatrix}$$

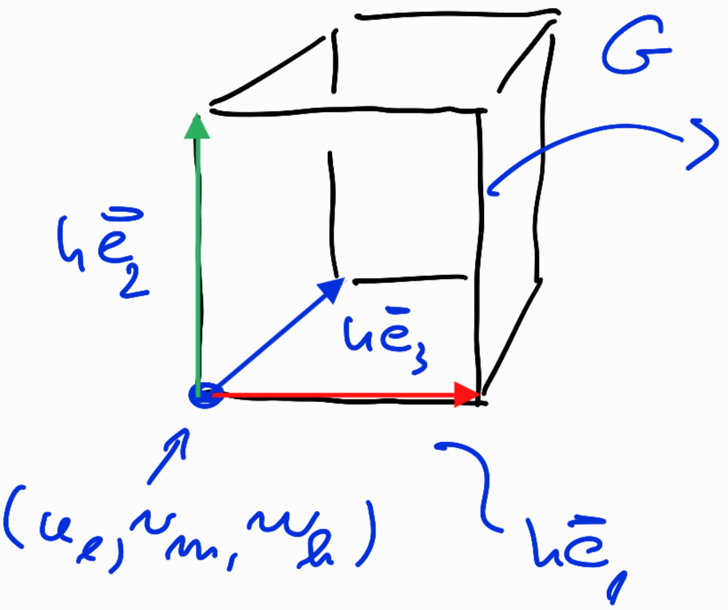
2) <u>Vollkugel</u> :

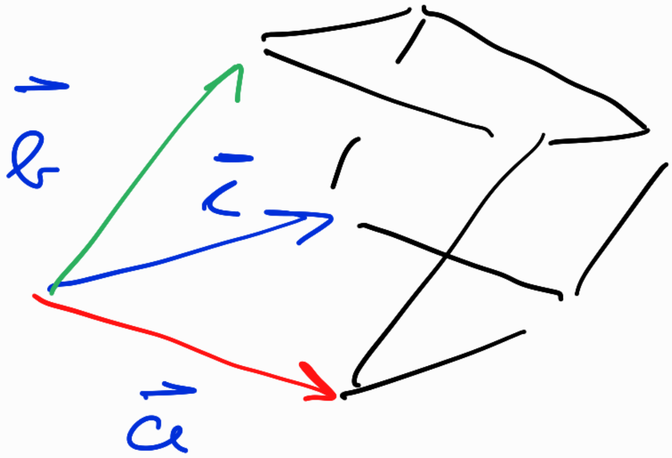
$$B : [0, \pi] \times [0, 2\pi] \times [0, R] \to \mathbb{R}^3$$

$$\underset{=}{(\vartheta}, \varphi, r) \mapsto \begin{pmatrix} r\cos\varphi \sin\vartheta \\ r\sin\varphi \sin\vartheta \\ r\cos\vartheta \end{pmatrix}$$



Volumenintegral





$h \to 0$

$$\Delta V = \langle \vec{a} \times \vec{b}, \vec{c} \rangle$$

$$\left.\begin{aligned} \vec{a} &= \frac{\partial \vec{G}}{\partial u}\, h \\ \vec{b} &= \frac{\partial \vec{G}}{\partial v}\, h \\ \vec{c} &= \frac{\partial \vec{G}}{\partial w}\, h \end{aligned}\right\} \to$$

$$\Delta V = \left\langle \frac{\partial \vec{G}}{\partial u} \times \frac{\partial \vec{G}}{\partial v}, \frac{\partial \vec{G}}{\partial w} \right\rangle h^3$$

$$\Rightarrow \int_G f\,dV = \sum_{l,m,h} f(\vec{G}_{l,m,h})\,\Delta V_{l,m,h}$$

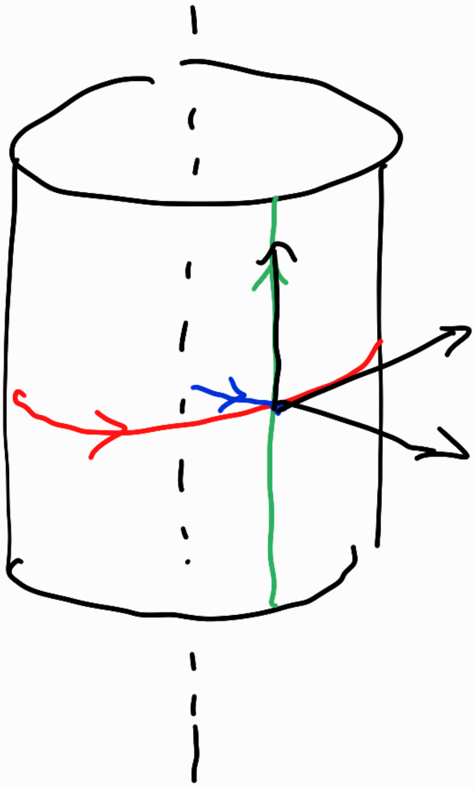
$$\boxed{\begin{gathered}\int_G f\,dV := \\ \int_{u_1}^{u_2}\int_{v_1}^{v_2}\int_{w_1}^{w_2} f(\vec{G})\left\langle \frac{\partial\vec{G}}{\partial u}\times\frac{\partial\vec{G}}{\partial v}, \frac{\partial\vec{G}}{\partial w}\right\rangle dw\,dv\,du\end{gathered}}$$

Volumeninhalt: $f = 1$

$$V(G) = \int_G 1\,dV$$

1) Vollzylinder:

$$\vec{Z}(\varphi, z, \rho) = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ z \end{pmatrix}$$



$$\frac{\partial \vec{Z}}{\partial \varphi} = \rho\,\vec{e}_\varphi$$

$$\frac{\partial \vec{Z}}{\partial z} = \vec{e}_z$$

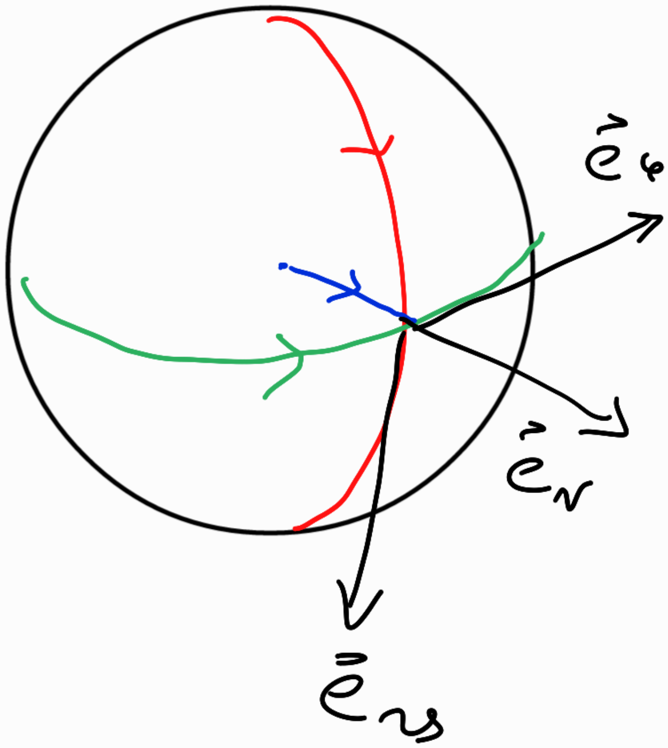
$$\frac{\partial \vec{Z}}{\partial \rho} = \vec{e}_\rho$$

$$\rightsquigarrow \left\langle \frac{\partial \vec{Z}}{\partial \varphi} \times \frac{\partial \vec{Z}}{\partial z}, \frac{\partial \vec{Z}}{\partial \rho} \right\rangle = \rho \qquad !$$

$$\int_{Z_{R,H}} f\,dV = \int_0^{2\pi}\int_0^H\int_0^R f(z,\rho,\varphi)\,\underbrace{\rho\,d\rho\,dz\,d\varphi}_{\text{Volumenelement}}$$

2) Vollkugel:

$$\vec{B}(\vartheta, \varphi, r) = \underline{r} \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix}$$



$$\frac{\partial \vec{B}}{\partial \vartheta} = \underline{r}\, \vec{e}_\vartheta$$

$$\frac{\partial \vec{B}}{\partial \varphi} = \underline{r \sin\vartheta}\, \vec{e}_\varphi$$

$$\frac{\partial \vec{B}}{\partial r} = \vec{e}_r$$

$$\Rightarrow \left\langle \frac{\partial \vec{B}}{\partial \vartheta} \times \frac{\partial \vec{B}}{\partial \varphi}, \frac{\partial \vec{B}}{\partial r} \right\rangle = \underline{\underline{r^2 \sin\vartheta}}\ !$$

$$\int_{B_R} f\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} f(\vartheta,\varphi,r)\,\cdot$$

$$r^2 \sin\vartheta \, d\vartheta \, d\varphi \, dr$$

Beispiele:

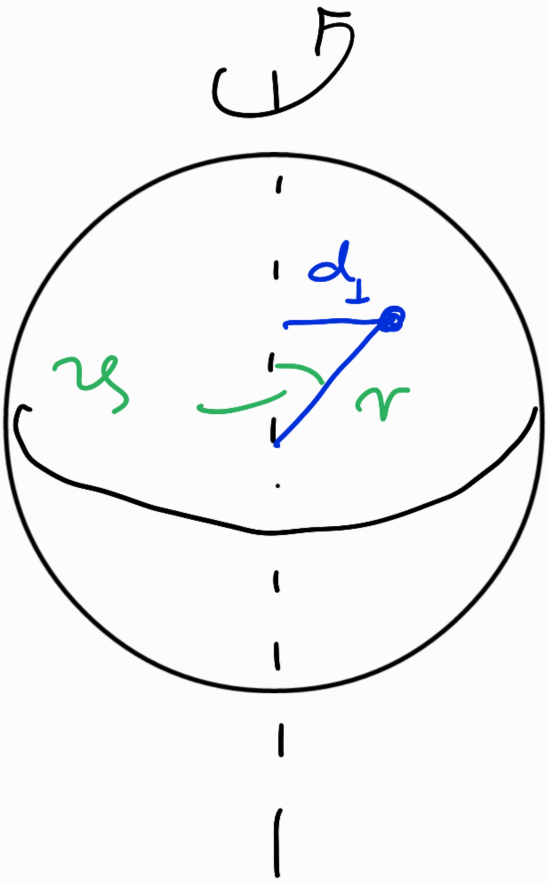
1) Kugelvolumen: $V(B_R) =$

$$\int_{B_R} 1\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} r^2 \sin\vartheta \, d\vartheta \, d\varphi \, dr$$

$$= \int_0^{\pi} d\vartheta \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \sin\vartheta \underbrace{\int_0^{R} r^2\,dr}_{= R^3/3}$$

$$= 2\pi \frac{R^3}{3} \Big(\underbrace{-\cos\vartheta \Big|_0^{\pi}}_{= 2}\Big) = \frac{4}{3}\pi R^3 \;!$$

2) Trägheitsmoment einer homogenen Kugel I:



$$\rho_0 = \frac{M}{\frac{4}{3}\pi R^3}$$

$$I = \int_{B_R} d_\perp^2 \, \rho_0 \, dV$$

$$d_\perp = r \sin\vartheta$$

$$I = \rho_0 \int_0^{\pi} d\vartheta \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \int_0^{R} dr \, \underbrace{(r \sin\vartheta)^2}_{= d_\perp^2} \underbrace{r^2 \sin\vartheta}_{\text{Volume-element}}$$

$$= \rho_0 \, 2\pi \int_0^{\pi} \sin^3\vartheta \, d\vartheta \, \underbrace{\int_0^{R} r^4 \, dr}_{R^5/5}$$

$$\left[\int_0^{\pi} \underbrace{\sin^3\vartheta}_{\sin\vartheta(1-\cos^2\vartheta)} d\vartheta = \underbrace{\int_0^{\pi} \sin\vartheta \, d\vartheta}_{=2} + \underbrace{\int_0^{\pi} \overbrace{(-\sin\vartheta)\cos^2\vartheta}^{\frac{d}{d\vartheta}\frac{\cos^3\vartheta}{3}} \, d\vartheta}_{-2/3} \right.$$

$$= -\cos\vartheta \Big|_0^{\pi} + \frac{\cos^3\vartheta}{3} \Big|_0^{\pi} = 4/3$$

$$\rightarrow \quad I = \rho_0 \, 2\pi \, \frac{4}{3} \, \frac{R^5}{5}$$

$$\rho_0 = \frac{M}{\frac{4}{3}\pi R^3}$$

$$\Rightarrow \quad \boxed{I = \frac{2}{5} M R^2}$$