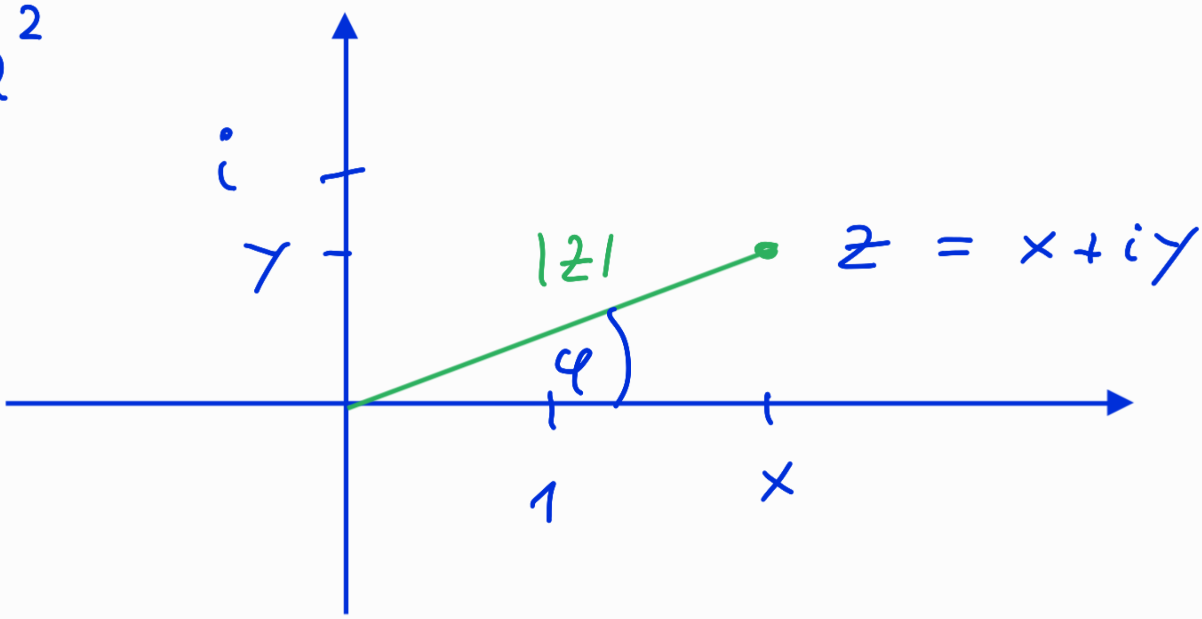
letzte Vrlsg.: <u>komplexe Zahlen</u>

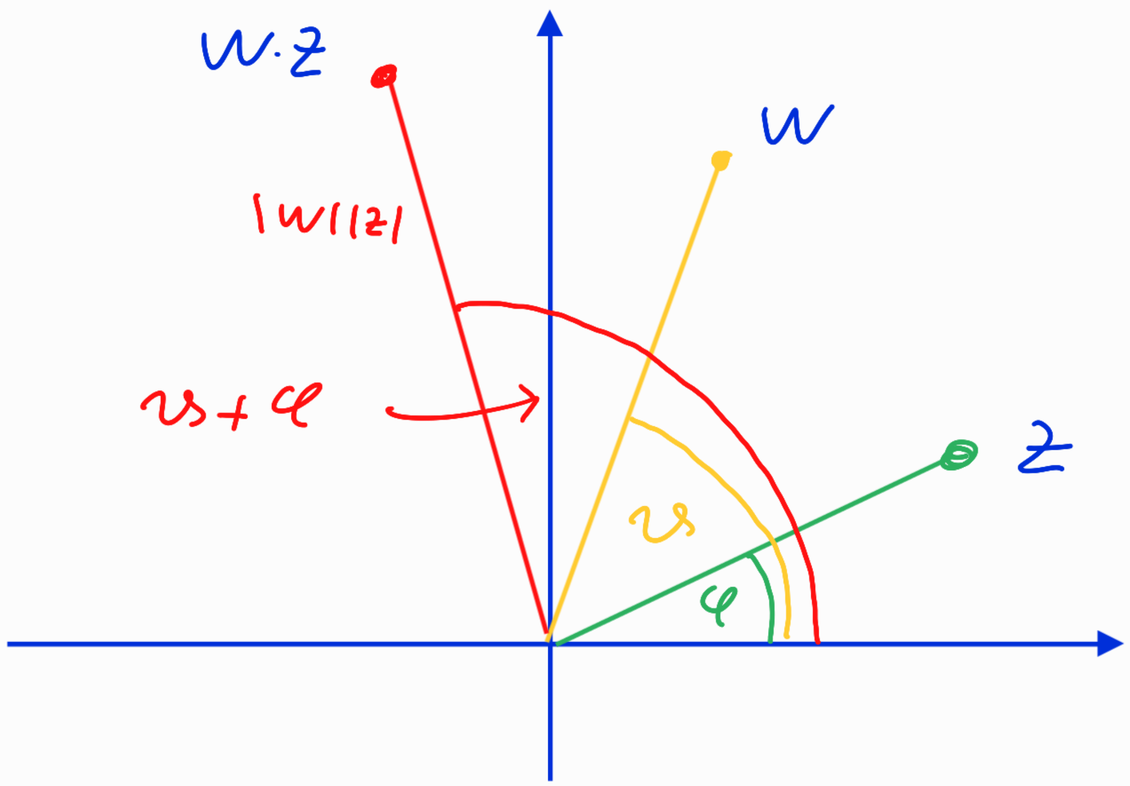
$\mathbb{C} = \mathbb{R}^2$



- $\operatorname{Re} z = x$
- $\operatorname{Im} z = y$
- $|z| = \sqrt{x^2 + y^2}$
- $\arg z = \varphi$

- <u>Addition</u> = Vektoraddition

- Multiplikation:



$$\Rightarrow \quad \boxed{i^2 = -1}$$

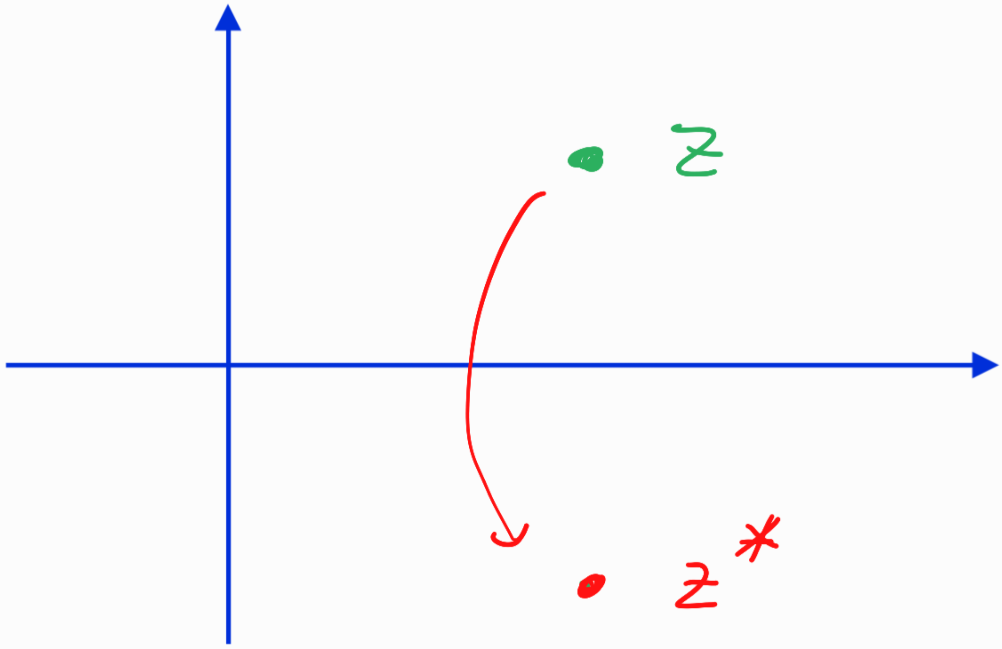
z.B.: $(a+ib)(x+iy)$

$= ax + i^2 by + iay + ibx$

$= ax - by + i(ay + bx)$

komplexe Konjugation

= Spiegelung an reeller Achse:



$$z = x + iy \xrightarrow{*} z^* = x - iy$$

$\rightsquigarrow$

- $\mathrm{Re}\, z = \frac{1}{2}(z + z^*)$
- $\mathrm{Im}\, z = \frac{1}{2i}(z - z^*)$
- $|z|^2 = z z^*$
- $z^{-1} \equiv \frac{1}{z} = \frac{z^*}{|z|^2}$

Nachfrag:

- $(w+z)^* = w^* + z^*$
- $(w z)^* = w^* z^*$
- $\left(\frac{1}{z}\right)^* = \frac{1}{z^*}$
- $(z^*)^* = z$

Taylorentw. von exp, sin, cos

$\longrightarrow$ Potenzreihen:

$$\exp x = \sum_{\ell=0}^{\infty} \frac{x^\ell}{\ell!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$\sin x = \sum_{\ell=0}^{\infty} (-1)^\ell \frac{x^{2\ell+1}}{(2\ell+1)!} = x - \frac{x^3}{6}$$

$$\cos x = \sum_{\ell=0}^{\infty} (-1)^\ell \frac{x^{2\ell}}{(2\ell)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

2) **komplexe Exponentialfunktion:**

$$\boxed{e^z \equiv \exp z = \sum_{\ell=0}^{\infty} \frac{z^\ell}{\ell!}}$$

($\sin z$, $\cos z$ analog)

$$\underbrace{e^{x+iy} = ?}_{} \qquad (x, y \in \mathbb{R})$$

$$\hookrightarrow e^x \, e^{iy} \quad ?$$

$$e^{iy} = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (iy)^\ell$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!} (iy)^{2\ell} + \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)!} (iy)^{2\ell+1}$$

$$i^{2\ell} = (-1)^\ell$$

$$\stackrel{\downarrow}{=} \underbrace{\sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell)!} y^{2\ell}}_{\cong \cos y} + i \underbrace{\sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)!} y^{2\ell+1}}_{\cong \sin y}$$

$$\rightsquigarrow \boxed{e^{iy} = \cos y + i \sin y}$$

$y \in \mathbb{R}$

Euler-Identität

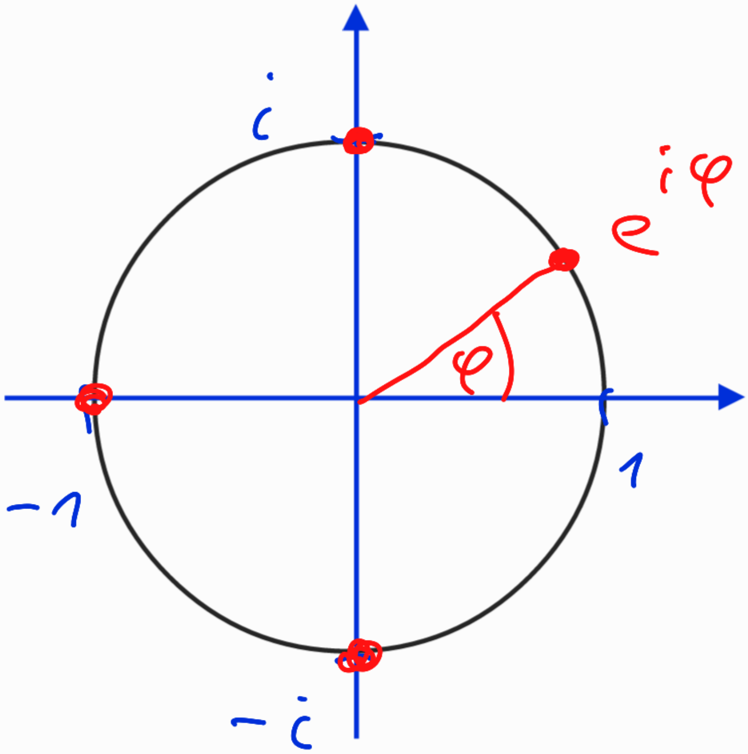
$$\boxed{\begin{aligned} \cos y &= \mathrm{Re}\, e^{iy} = \frac{1}{2}\left(e^{iy} + e^{-iy}\right) \\ \sin y &= \mathrm{Im}\, e^{iy} = \frac{1}{2i}\left(e^{iy} - e^{-iy}\right) \end{aligned}}$$

$$|e^{iy}| = 1$$

$$\left[\; |e^{iy}|^2 = e^{iy} \cdot \left(e^{iy}\right)^* = e^{iy} e^{-iy} = 1 \;\right]$$

Euler Identität geometrisch:

$$\boxed{e^{i\varphi} = \cos\varphi + i\sin\varphi} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$



$\rightarrow \quad i = e^{i\pi/2}$

$-1 = e^{i\pi}$

$-i = e^{i\,3\pi/2}$

$1 = e^{i0} = e^{i2\pi} = e^{i4\pi}$

Polardarstellung:

$$\boxed{z = |z|\, e^{i\varphi}} \qquad \varphi = \arg z$$

<u>Anwendungsbeispiele</u>

1) Lösungen der Gl. $\boxed{z^n \overset{!}{=} 1}$

→ n-te Einheitswurzeln W_n ?

Ansatz: $\quad z = e^{i\varphi}$

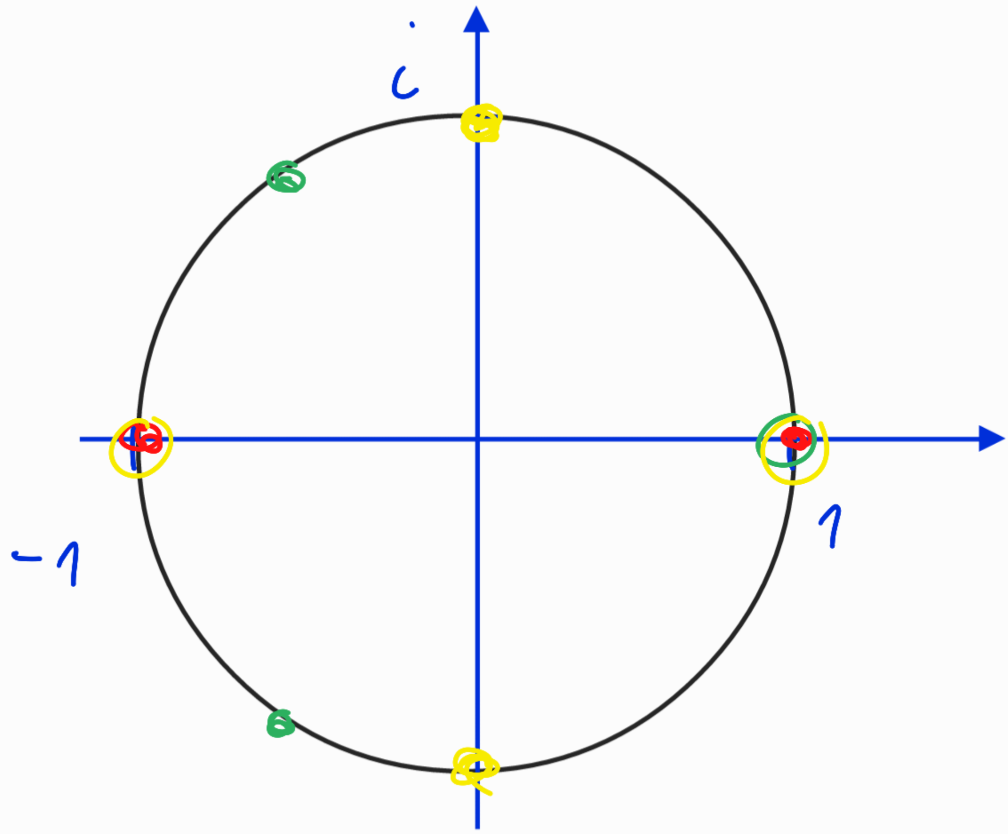
$$\to \quad z^n = (e^{i\varphi})^n = e^{i\varphi n} \overset{!}{=} 1$$

d.h. $\quad \varphi n = 2\pi k\,, \quad k \in \mathbb{Z}$

$$\leadsto \quad \varphi = \frac{2\pi k}{n} \quad !$$

$$\leadsto \quad W_n \doteq \left\{ e^{i\frac{2\pi k}{n}} \;\middle|\; k = 0, 1, 2, \ldots, n-1 \right\}$$

$$W_n = \left\{ e^{i\frac{2\pi k}{n}} \;\middle|\; k = 0, 1, \dots, n-1 \right\}$$



$$W_2 = \{1, -1\} = \{e^0, e^{\pi i}\}$$

$$W_3 = \left\{1, e^{i\frac{2\pi}{3}}, e^{i\,4\pi/3}\right\}$$

$$W_4 = \left\{1, e^{i\pi/2}, e^{i\pi}, e^{i\frac{3\pi}{2}}\right\}$$

2) Additionstheoreme für sin und cos:

$$\underline{\underline{\cos(\varphi+\vartheta)}} + i\, \underline{\underline{\sin(\varphi+\vartheta)}}$$

$$\overset{E.I.}{=} e^{i(\varphi+\vartheta)}$$

$$= e^{i\varphi} \cdot e^{i\vartheta}$$

$$\overset{E.I.}{=} (\cos\varphi + i\sin\varphi)\cdot(\cos\vartheta + i\sin\vartheta)$$

$$= \underline{\underline{(\cos\varphi\cos\vartheta - \sin\varphi\sin\vartheta)}}$$

$$+ i\,\underline{\underline{(\cos\varphi\sin\vartheta + \sin\varphi\cos\vartheta)}}$$