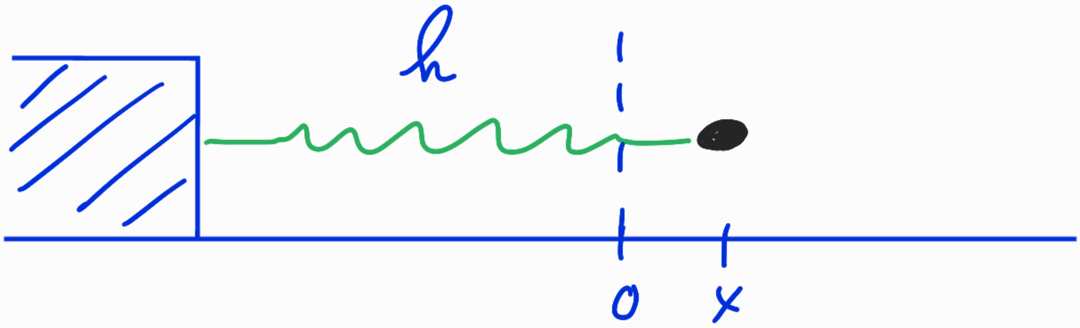
letzte Vrlsg.:

Harmonischer Oszillator mit Dämpfung



- Federkraft: $F(x) = -k\underline{x}$
- Reibungskraft: $F_r(\dot{x}) = -\gamma m \underline{\dot{x}}$

⇒ Newtonsche Bewegungsgl.:

$$\boxed{\ddot{x} + \gamma \dot{x} + \frac{k}{m} x = 0}$$

↳ DGL 2. Ordnung, linear, homogen

- Lösung mittels <u>Exponentialansatz</u>

$$x(t) = e^{\lambda t}$$

$\rightsquigarrow$ λ bestimmt durch

$$\boxed{\lambda^2 + \gamma\lambda + \frac{k}{m} \overset{!}{=} 0}$$

$$\rightsquigarrow \quad \lambda_{1/2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \frac{k}{m}}$$

$\rightsquigarrow$ Lsg.en $\quad x_{1/2}(t) = e^{\lambda_{1/2} t}$

Fallunterscheidung:

a) keine Dämpfung: $\gamma = 0$

b) schwache " " : $\gamma^2/4 < k/m$

c) starke " " : $\gamma^2/4 > k/m$

d) Grenzfall : $\gamma^2/4 = k/m$

a) $\underline{\gamma = 0}$: $\boxed{\lambda_{1/2} = \pm \underline{\underline{i}}\, \omega_0}$

$$\omega_0 = \sqrt{k/m}$$

→ <u>reelle</u> Lösungen:

$$\tilde{x}_1(t) = \mathrm{Re}\, e^{i\omega_0 t} = \cos \omega_0 t$$

$$\tilde{x}_2(t) = \mathrm{Im}\, e^{i\omega_0 t} = \sin \omega_0 t$$

→ allg. (reelle) Lsg.:

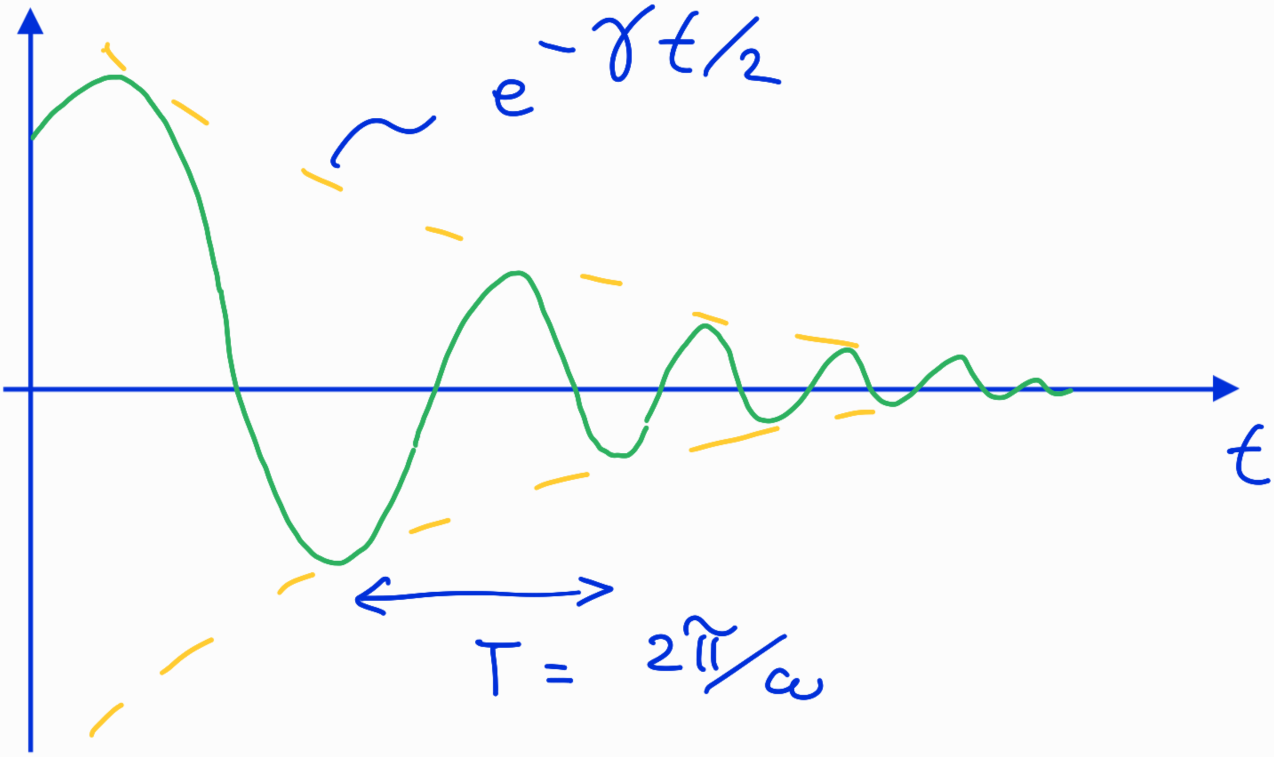
$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

b) <u>schwache Dämpfung</u>: $\gamma^2/4 < k/m$

$$\leadsto \quad \boxed{\lambda_{1/2} = -\underline{\underline{\frac{\gamma}{2}}} \pm \underline{\underline{i}}\, \omega}$$

$$\omega = \sqrt{k/m - \gamma^2/4} \qquad (< \omega_0)$$

$$\rightarrow \quad x(t) = e^{-\gamma t/2}\left(c_1 \cos \omega t + c_2 \sin \omega t\right)$$



„gedämpfte Schwingung"

1) Starke Dämpfung: $\frac{\gamma^2}{4} > k/m$

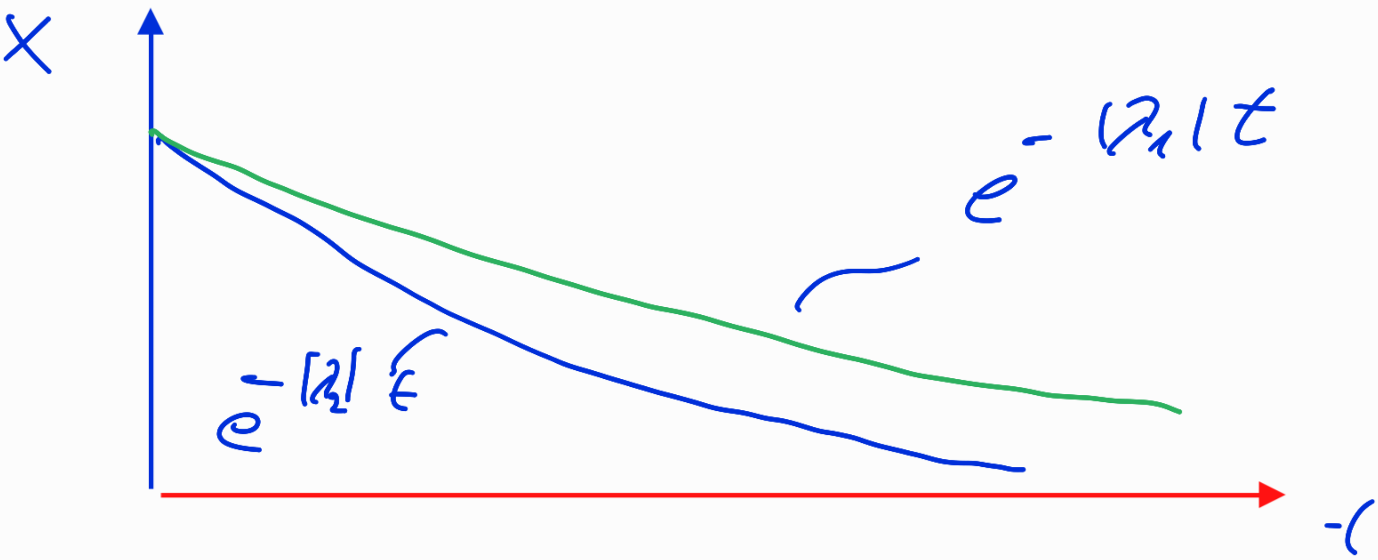
$\rightsquigarrow$ reelle Nullstellen

$$\lambda_{1/2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - k/m}$$

- $\lambda_2 < \lambda_1 < 0$

$\rightsquigarrow$ allg. Lsg.:

$$\boxed{x(t) = c_1 e^{-|\lambda_1| t} + c_2 e^{-|\lambda_2| t}}$$



d) <u>Grenzfall</u> : $\frac{\gamma^2}{4} = \frac{k}{m}$

$$\rightarrow \quad \ddot{x} + \gamma \dot{x} + \frac{\gamma^2}{4} x = 0$$

$$x(t) = e^{\lambda t}$$

$$\lambda^2 + \gamma \lambda + \frac{\gamma^2}{4} x = 0$$

→ <u>doppelte</u> Nullstelle

$$\boxed{\lambda_1 = \lambda_2 = -\frac{\gamma}{2}}$$

$\Rightarrow$ Lsg $x(t) = e^{-\gamma t/2}$ ✓

2. Lösung unabhängig von

?

DGL: $$\boxed{\ddot{x} + \gamma \dot{x} + \frac{\gamma^2}{4} x = 0}$$

$\frac{\gamma^2}{4} = \frac{k}{m}$

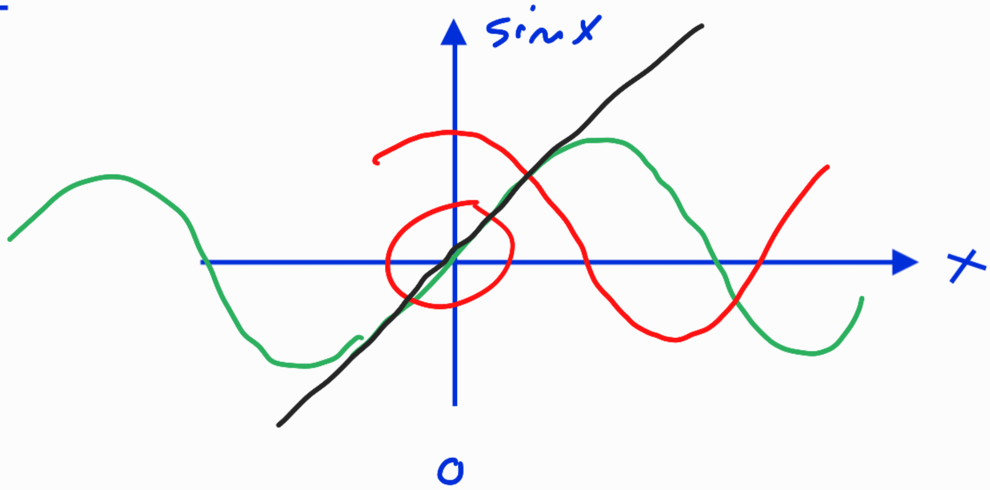
Betrachte gedämpfte Schwingung im Grenzfall

$$\omega = \sqrt{k/m - \gamma^2/4} \longrightarrow 0 + :$$

$$x(t) = e^{-\gamma t/2}\left(c_1 \underbrace{\cos(\omega t)}_{\overset{\omega t \to 0}{=} 1 - \cancel{\frac{1}{2}(\omega t)^2}} + c_2 \underbrace{\sin(\omega t)}_{= \omega t - \cancel{\frac{(\omega t)^3}{6}}}\right)$$

$\rightarrow$ 2. Lösung: $$\boxed{x_2(t) = t\, e^{-\gamma t/2}}$$

?



$$\sin x = x - \frac{1}{6}x^3$$

$$\cos x = 1 - \frac{1}{2}x^2$$

<u>Test</u>:

$$\frac{\gamma^2}{4}\; x_2(t) = t\, e^{-\gamma t/2}\; \gamma^2/4$$

$$\gamma\; \dot{x}_2(t) = \left(- \frac{\gamma^2}{2} t + 1\gamma\right) e^{-\gamma t/2}$$

$$+ \quad \ddot{x}_2(t) = \left(+ \frac{\gamma^2}{4} t - \gamma\right) e^{-\gamma t/2}$$

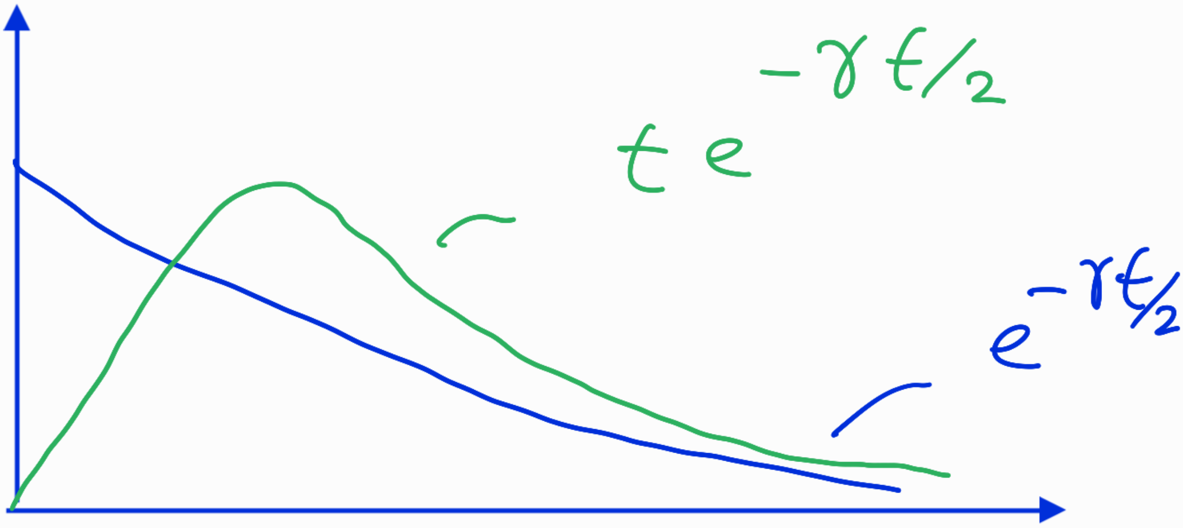
$$0 = 0 \quad !$$

✓

<u>allg. Lsg</u>:

allg. Lsg.: (Grenzfall)

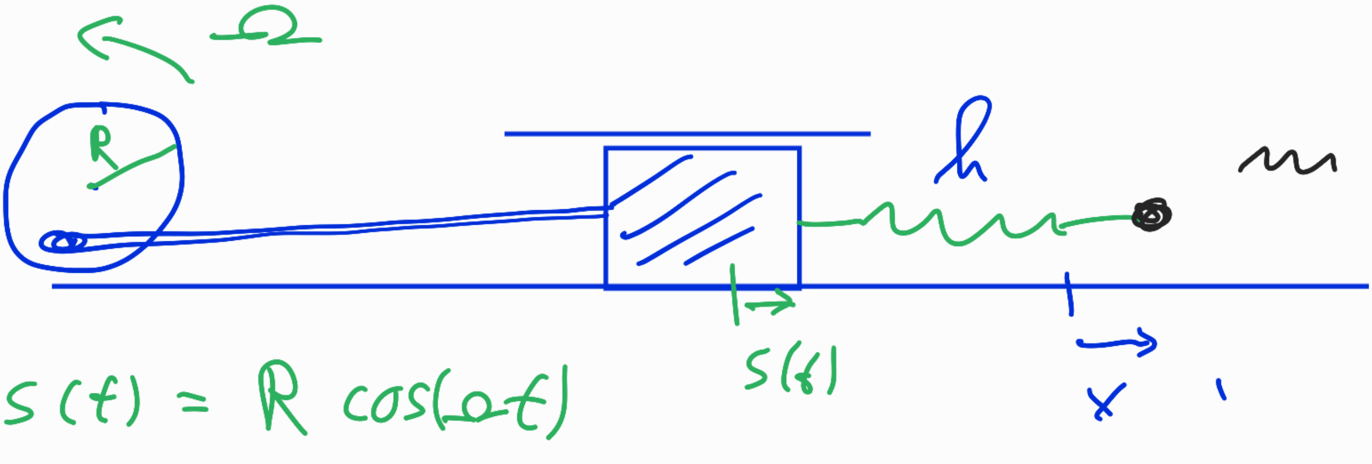
$$\boxed{x(t) = e^{-\gamma t/2}(\alpha + \beta t)}$$



Harmonischer Oszillator mit Dämpfung und periodischer externer Kraft, Resonanz

$$\boxed{F_{ex}(t) = F_0 \cos(\Omega t)}$$

ext. Freq. Ω



$$s(t) = R\cos(\Omega t)$$

$$F(x, s(t)) = -k(x - s(t))$$

$$= -kx + \underbrace{kR\cos(\Omega t)}_{\equiv F_{ex}(t)}$$

→ Newtonsche Bwg. gl.:

$$m\ddot{x} = -kx - \gamma m\dot{x} + F_0\cos(\Omega t)$$

(*)

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega^2 x = f_0\cos(\Omega t)}$$

$\omega = \sqrt{k/m}$

$f_0 = F_0/m$

DGL 2. Ordnung, linear, inhomogen!

allg. Lsg.:

$$x(t) = x_a(t) + x_s(t) \quad !$$

x_a: allg. Lsg. der homogenen Gl. ($f_0 = 0$)

$x_s(t)$: spez. Lsg. der inhomogenen DGL !

Problem: finde $x_s(t)$!

1) rechne komplex:

$$x_s(t) = \operatorname{Re} z_s(t)$$

wobei $z_s(t)$ komplexe Lsg der

DGL: $$\boxed{\ddot z + \gamma \dot z + \omega^2 z = f_0 e^{i\Omega t}} \quad (**)$$

$\rightsquigarrow$ $x_s(t) = \operatorname{Re} z_s(t)$ Lsg. der

DGL (*) ✓

2) Ansatz: $$\boxed{z_s(t) = a\, e^{i\Omega t}} ; \quad a \in \mathbb{C}$$

erzwungene Schwingung:

$$\boxed{\begin{array}{ll} \bullet \text{ Frequenz} & \Omega \\ \bullet \text{ Amplitude} & |a| \\ \bullet \text{ Phasenverschiebung} & -\varphi \end{array}}$$

$a = |a| e^{i\varphi}$

$z_s(t) = |a| e^{i(\Omega t + \varphi)}$

$\downarrow$

$x_s(t) = |a| \cos(\Omega t + \varphi)$

$z_s(t)$ eingesetzt in (**) :

$$z_s(t) = a\, e^{i\Omega t}$$

$$\dot{z}_s(t) = i\Omega\, a\, e^{i\Omega t}$$

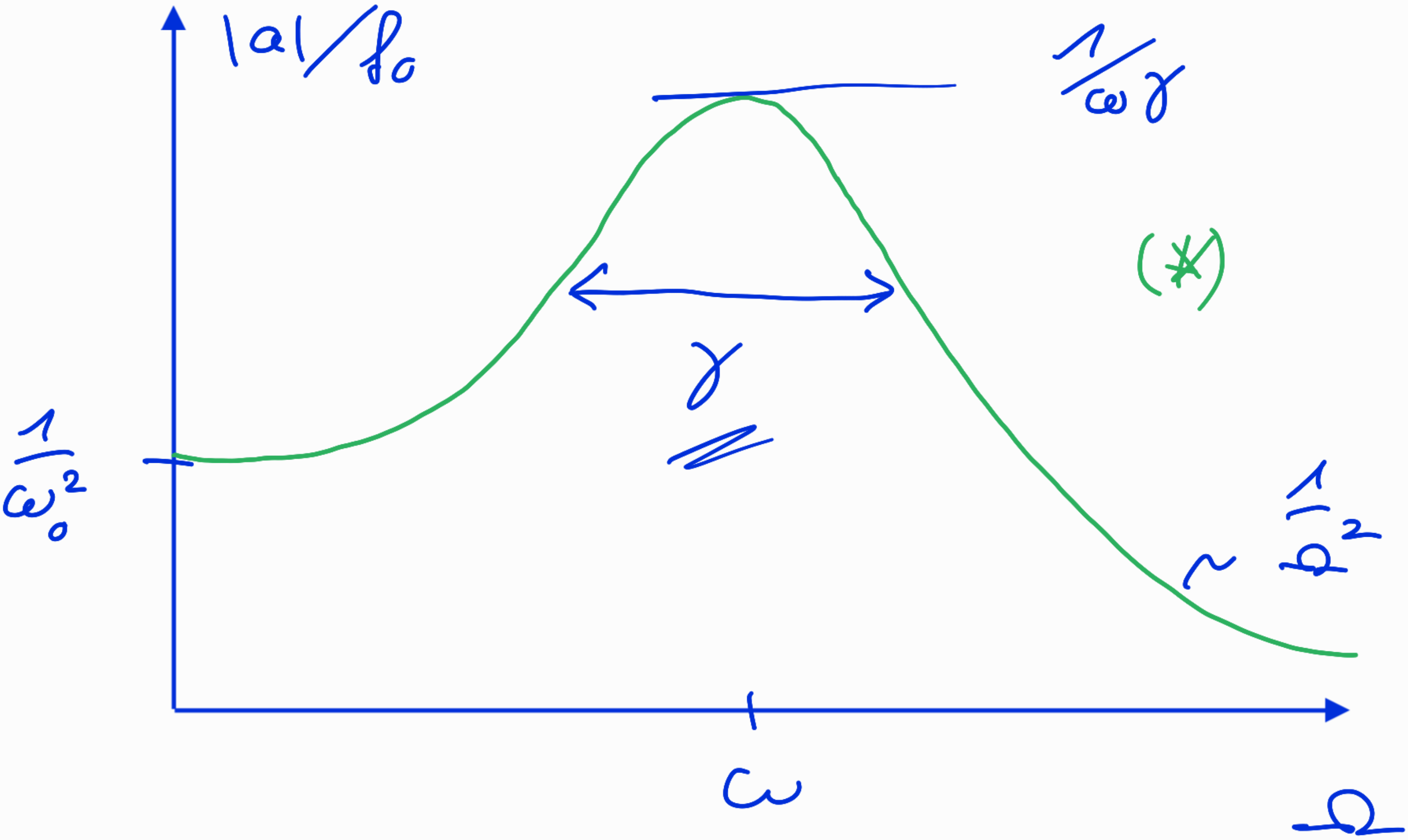
$$\ddot{z}_s(t) = -\Omega^2\, a\, e^{i\Omega t}$$

$$\rightarrow \quad \boxed{(-\Omega^2 + i\Omega\gamma + \omega^2)\, a = f_0}$$

$$\rightarrow \quad \boxed{a = \frac{f_0}{\omega^2 - \Omega^2 + i\Omega\gamma}}$$

Amplitude:

$$\boxed{|a| = \frac{f_0}{\sqrt{(\omega^2 - \Omega^2)^2 + \Omega^2\gamma^2}}}$$



$$(\omega^2-\Omega^2)^2 = \Big((\omega-\Omega)\underbrace{(\omega+\Omega)}_{\approx\, 2\Omega}\Big)^2$$

(*) "Resonanz - Peak"

- ω
- γ
- Ω

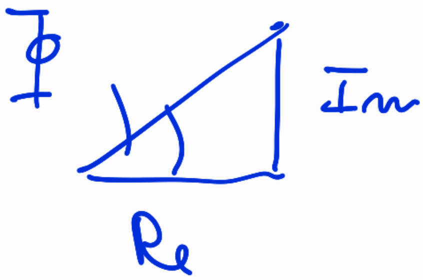
Phasenverschiebung:

Phasenverschiebung:

$$\underline{\underline{\vartheta}} = -\varphi = -\arg a$$

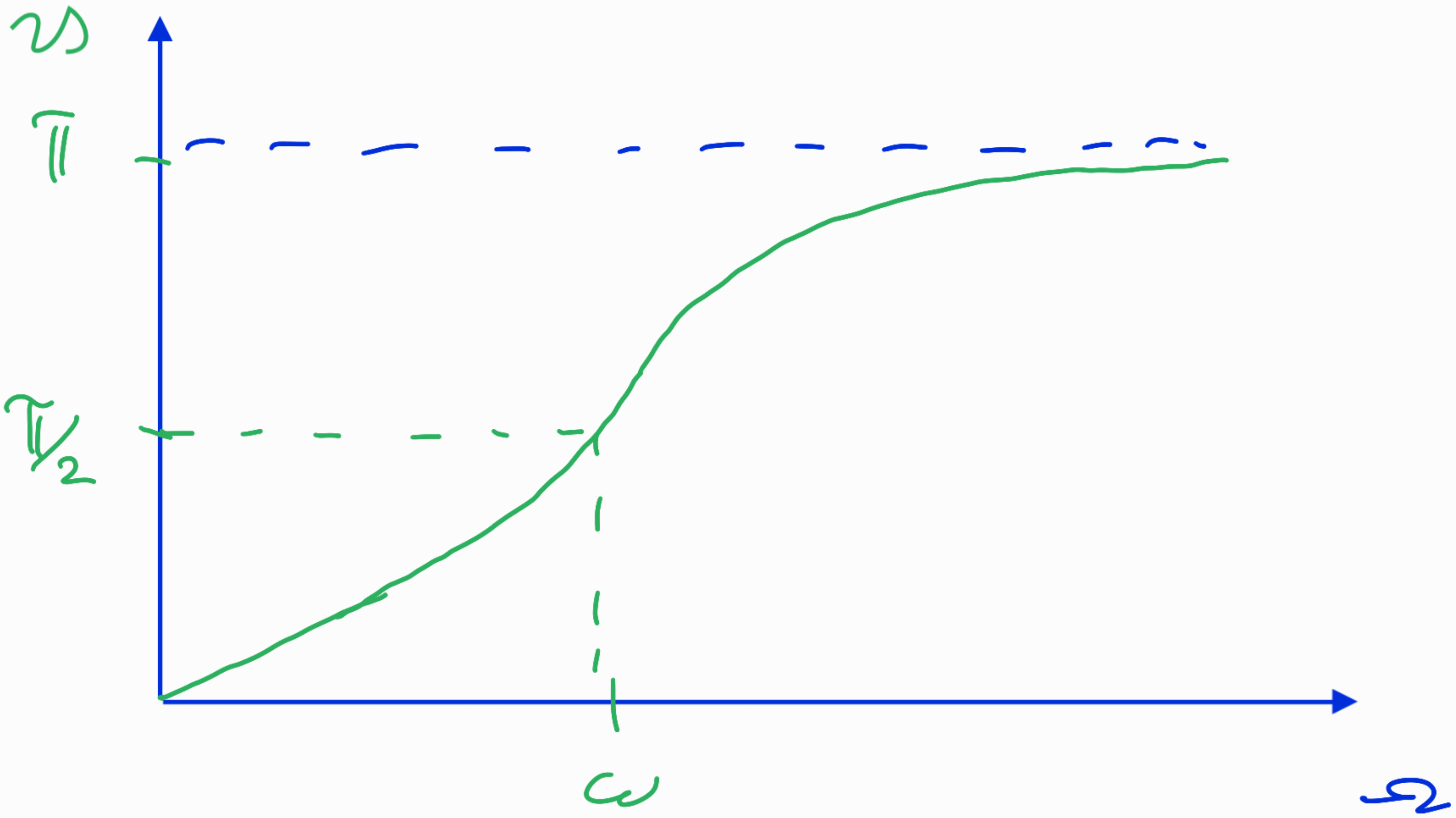
$$a = \frac{f_0}{\omega^2 - \Omega^2 + i\,\Omega\gamma}$$

$$\arg a = -\operatorname{atan} \frac{\Omega\gamma}{\omega^2 - \Omega^2}$$



$$\Phi = \operatorname{atan} \frac{\mathrm{Im}}{\mathrm{Re}}$$

$$\boxed{\vartheta = \operatorname{atan} \frac{\Omega\gamma}{\omega^2 - \Omega^2}}$$



ϑ
π
$\pi/2$
ω
Ω