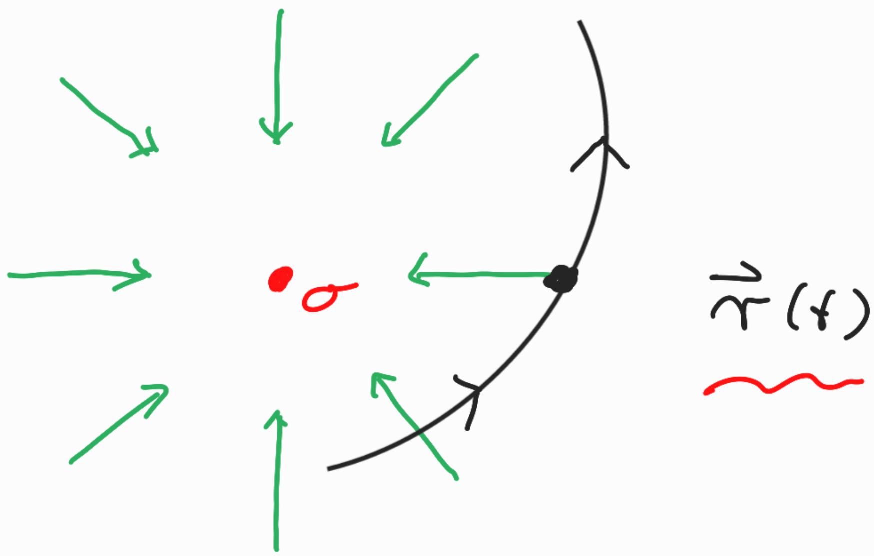
Letzte Vrlsg.:

Bwg. im isotropen Zentralkraftfeld



- $\vec{F}(\vec{r}) = g(r)\,\hat{r}$

- konservativ: $U(\vec{r}) = -G(r)$

  $(G' = g)$

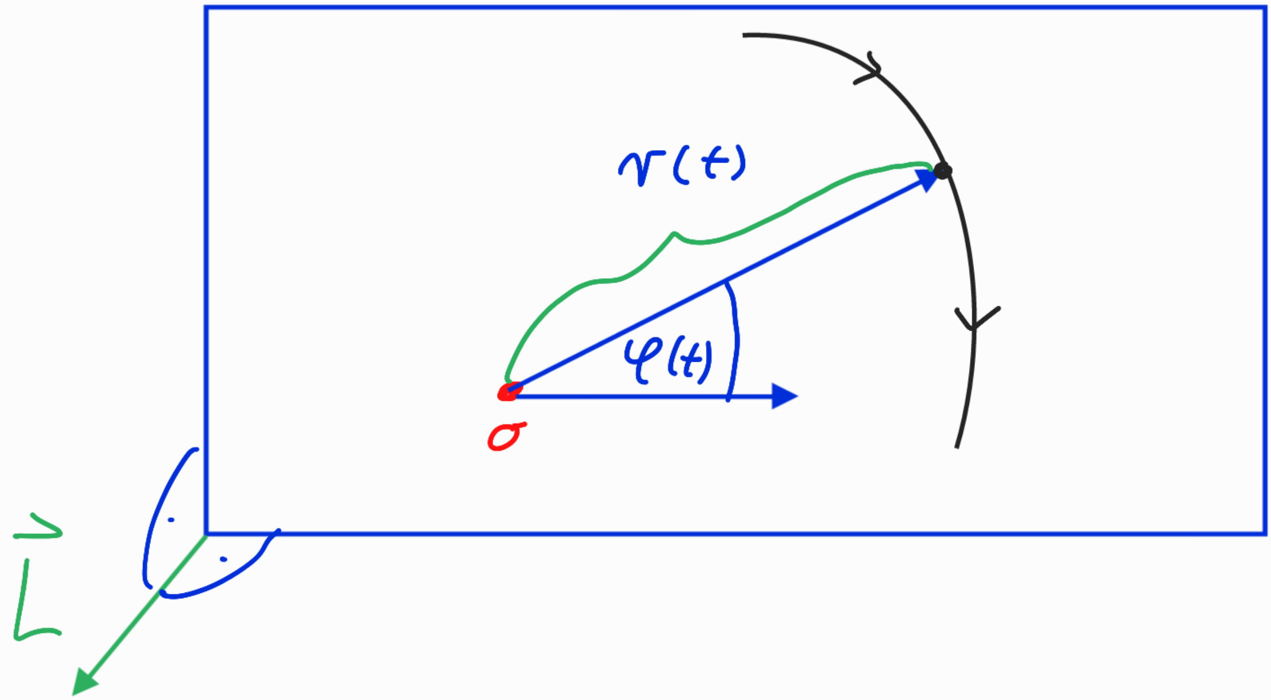
- $\vec{F} \parallel \vec{r} \rightarrow$ Drehimpulserhaltung

  $\rightarrow$ ebene Bewegung

  $\rightarrow$ Flächensatz

- $\vec{F}$ kons. $\rightarrow$ Energieerhaltung

• $\vec{r}(t) \longrightarrow$ <u>Polarkoordinaten</u>:



• $\vec{L} = $ konst:

$$\boxed{\dot{\varphi}(t) = \frac{\ell}{m\, r(t)^2}}$$

<u>Azimuthalgl.</u>, $\ell = |\vec{L}|$

• $$\boxed{E = \frac{m}{2}\dot{r}^2 + \underbrace{U(r) + \frac{\ell^2}{2 m r^2}}_{=:\, U_{eff}(r)}} = \text{konst}$$

<u>Radialgleichung</u> $\Rightarrow r(t) \longrightarrow \varphi(t)$

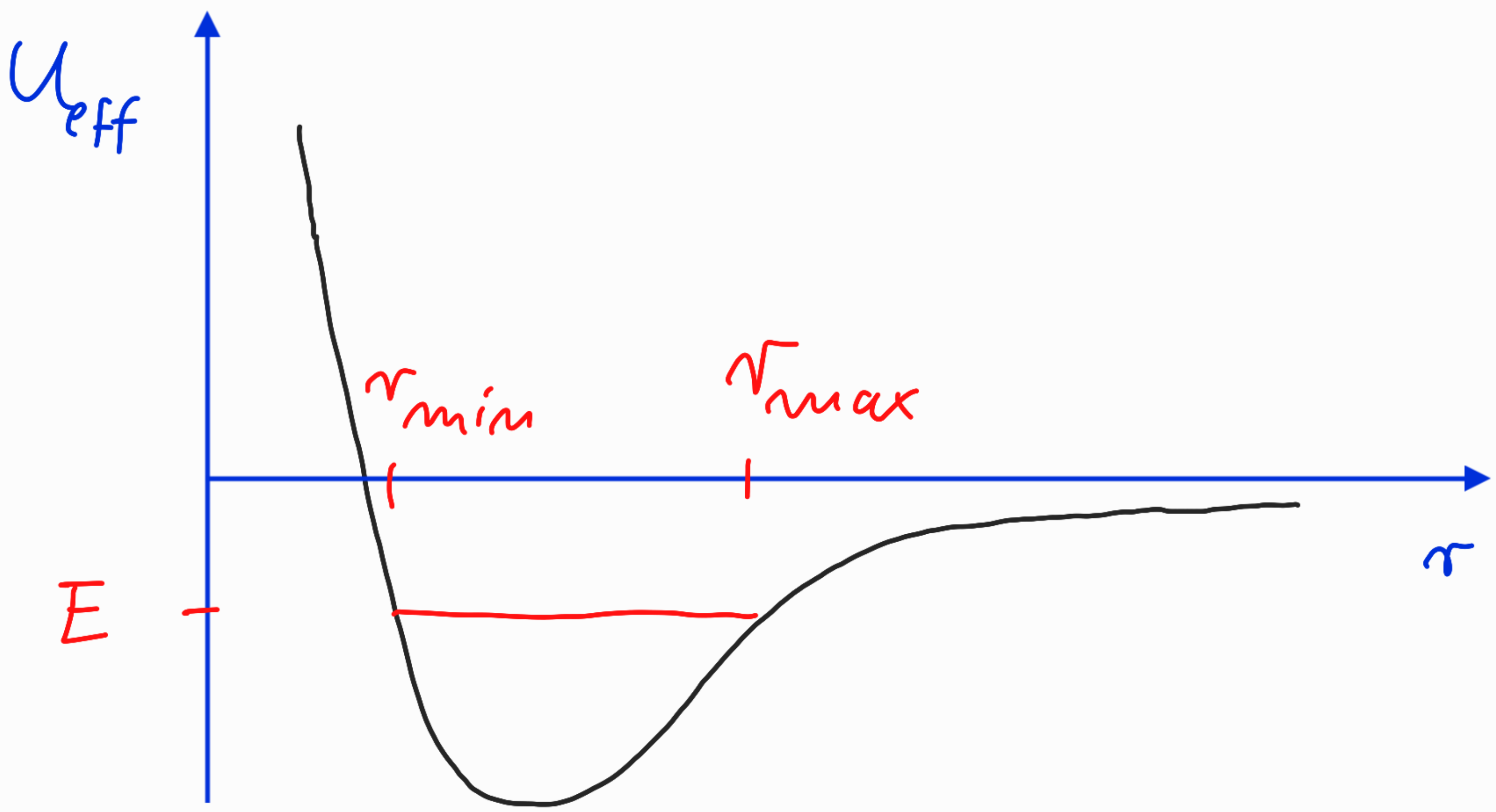
Beispiele:

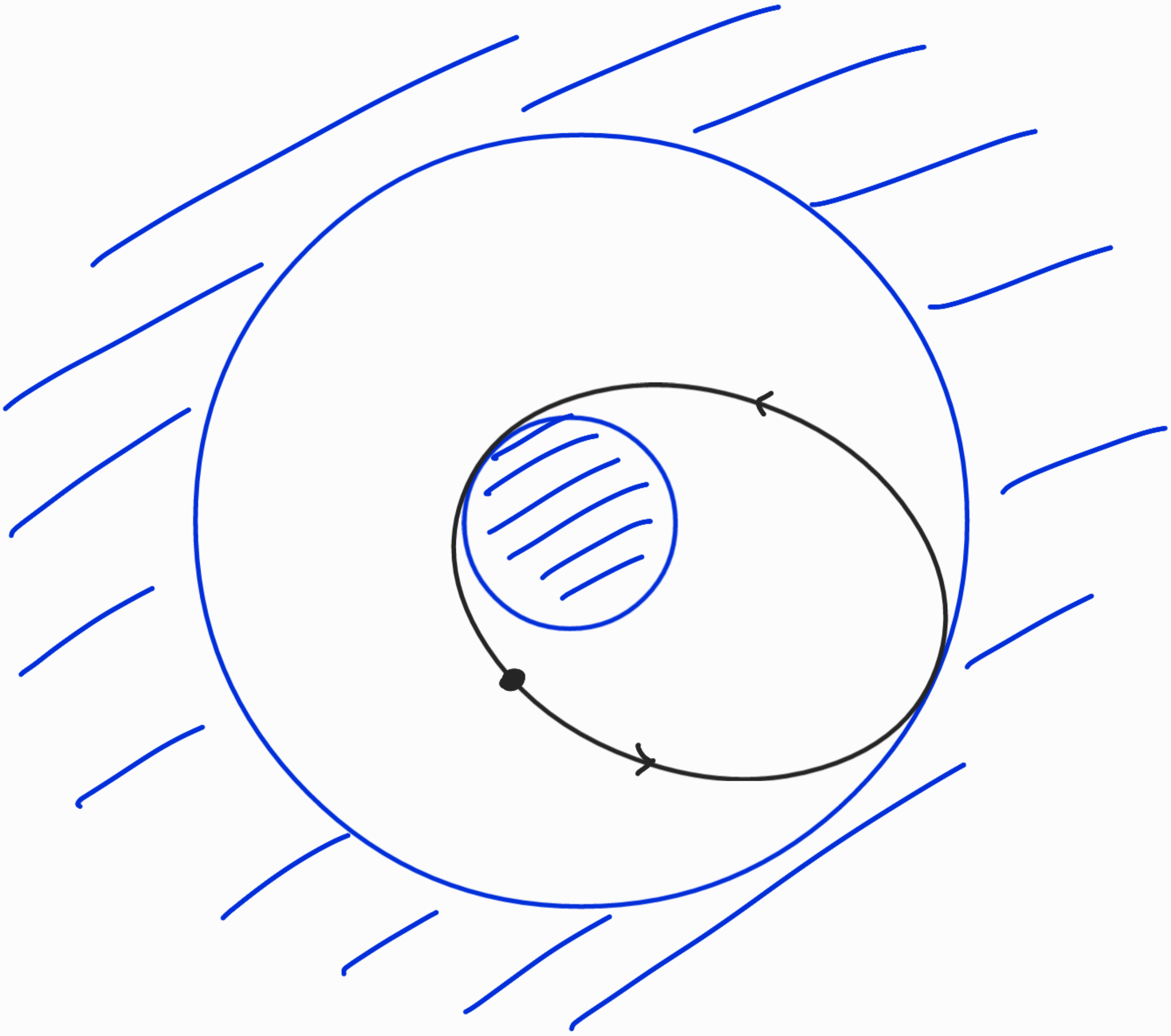
1) Planet im Gravitationsfeld der Sonne:

$$\vec{F}(r) = -\frac{\gamma}{r^2}\,\hat{r} \qquad (\gamma = GmM)$$

$$\rightarrow \quad U(r) = -\frac{\gamma}{r}$$

$$\rightarrow \quad U_{eff}(r) = -\frac{\gamma}{r} + \frac{\ell^2}{2mr^2}$$



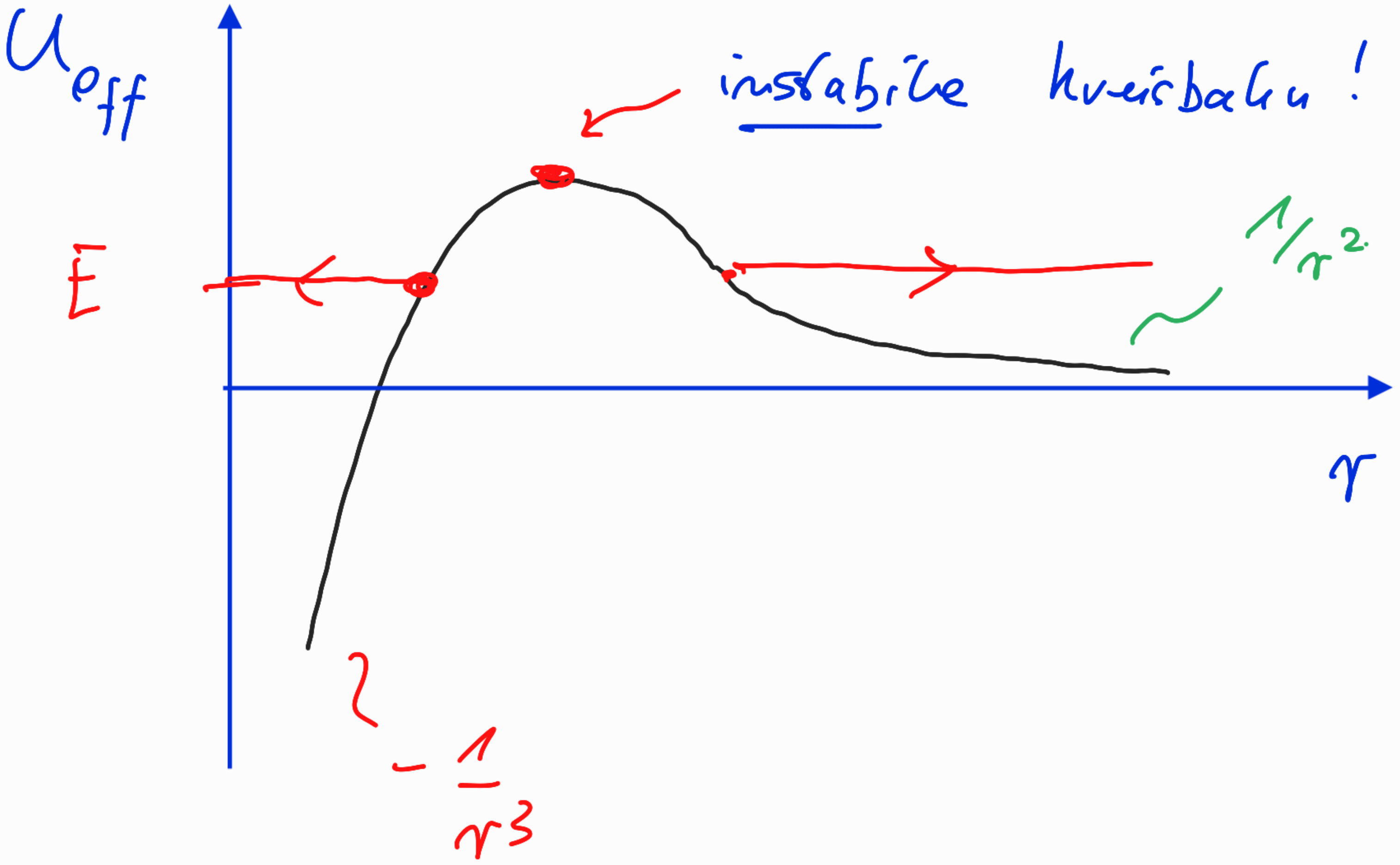


2) Teilchen im Kraftfeld

$$\vec{F}(\vec{r}) = -3\alpha \frac{1}{r^4} \hat{r}$$

$$\rightarrow \quad U(r) = -\frac{\alpha}{r^3}$$

$$\rightarrow \quad U_{eff}(r) = -\frac{\alpha}{r^3} + \frac{\ell^2}{2mr^2}$$



a : große Halbachse

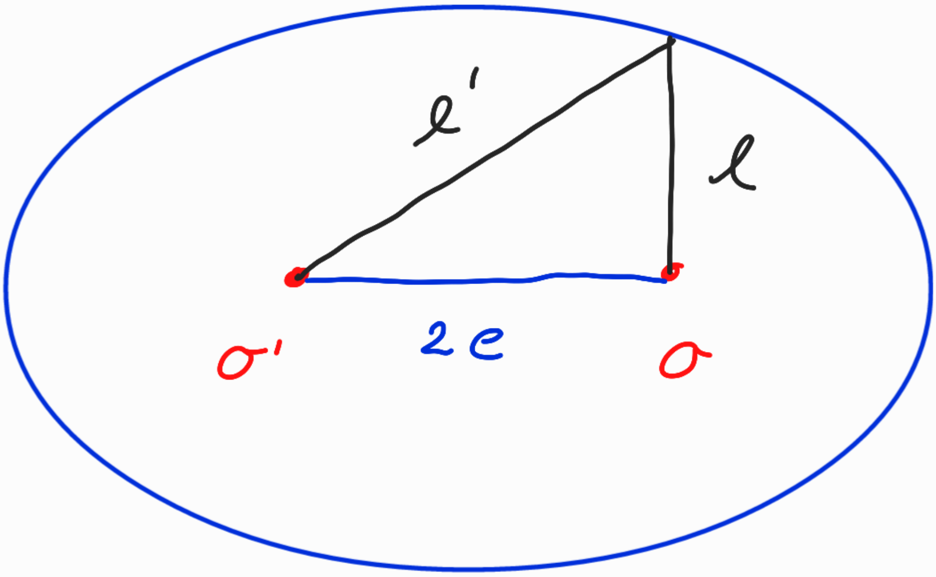
b : kleine " "

z : Zentrum

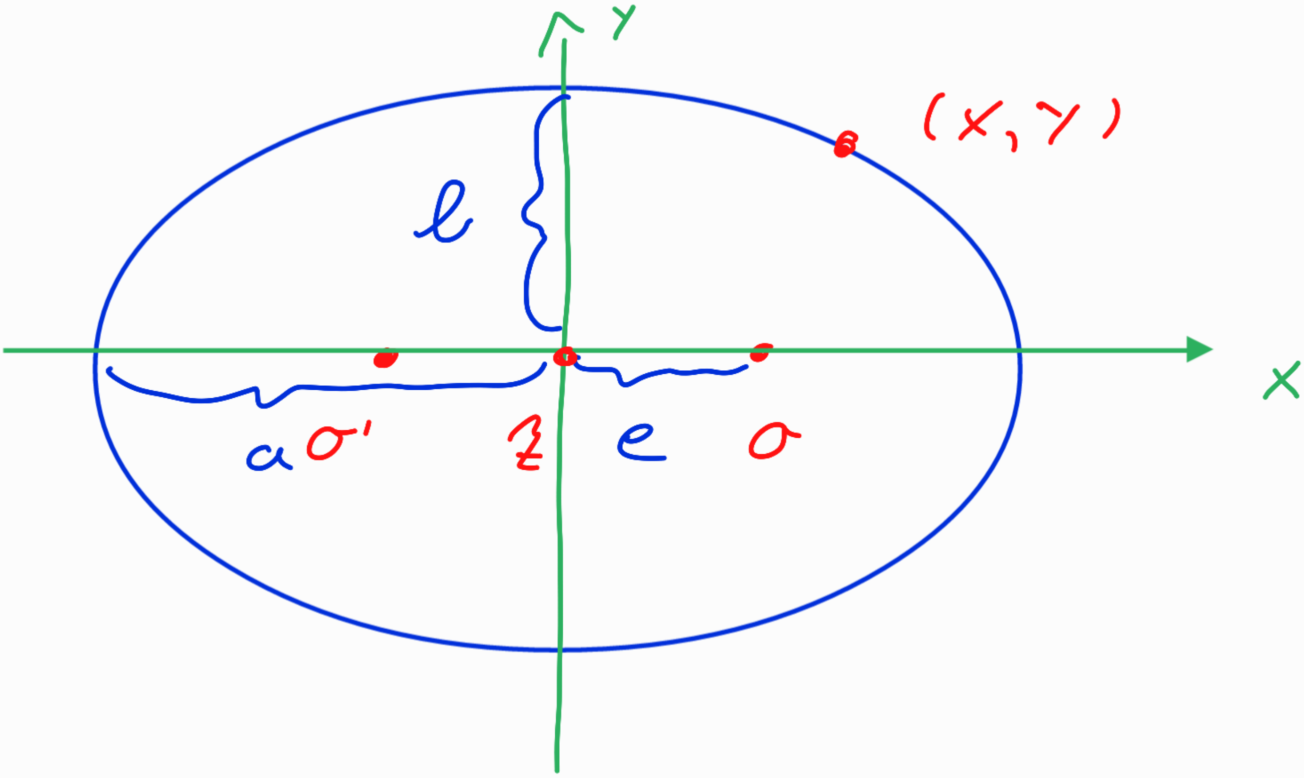
o, o' : Brennpunkte

$\varepsilon = \frac{e}{a}$: Exzentrizität

Exkurs: Darstellungen einer Ellipse



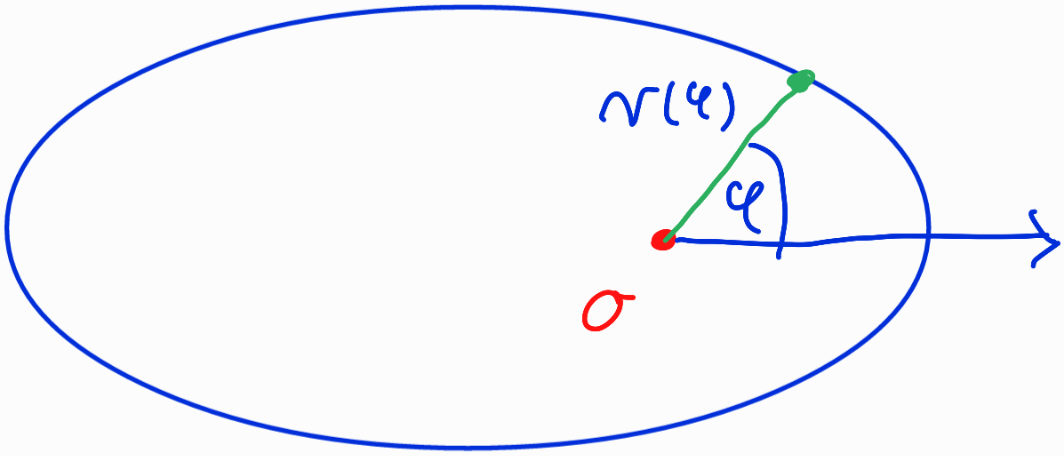
$$\boxed{\ell' + \ell \stackrel{!}{=} 2a}$$



$$b^2 = a^2 - e^2 = (1 - \varepsilon^2)\, a^2$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

Polardarstellung



$$\boxed{r(\varphi) = \frac{p}{1 + \varepsilon \cos \varphi}} \quad !$$

$\varepsilon = 0$: Kreis

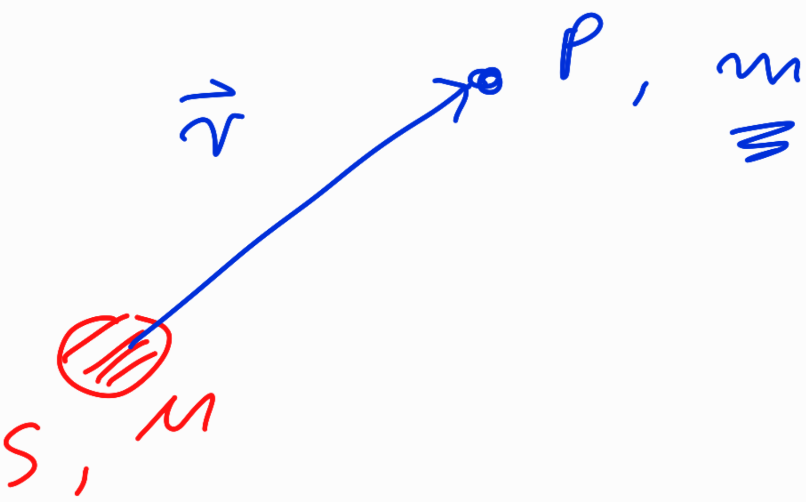
$0 < \varepsilon < 1$: Ellipse

($\varepsilon = 1$: Parabel)

($\varepsilon > 1$: Hyperbel)

$$p = (1 - \varepsilon^2)\, a \quad .$$

Bestimmung der Planetenbahnen:



$$\vec{F}(\vec{r}) = -\alpha \frac{m}{r^2}\,\hat{r} \quad , \quad \alpha = GM$$

$$U(r) = -\alpha \frac{m}{r}$$

$$U_{eff} = -\frac{\alpha m}{r} + \frac{\ell^2}{2mr^2} =$$

$$\lambda = \ell / m$$

$\lambda = \ell / m$.

$$U_{eff}(r) = -\frac{\alpha m}{r} + \frac{\lambda^2 m}{2 r^2}$$

$$\rightarrow \quad \frac{2E}{m} = \boxed{\dot{r}^2 - 2\frac{\alpha}{r} + \frac{\lambda^2}{r^2} =: c}$$

konst.!

$$\Rightarrow \quad \dot{r} = \sqrt{c + 2\frac{\alpha}{r} - \frac{\lambda^2}{r^2}} \qquad (*)$$

$$\Rightarrow r(t) \; !?$$

besser: $r(\varphi(t))$

$$\frac{dr}{dt} = \frac{dr}{d\varphi} \cdot \dot{\varphi} = \frac{\lambda}{r^2} \frac{dr}{d\varphi}$$

$$\dot{\varphi} = \frac{\ell}{m r^2} = \frac{\lambda}{r^2}$$

$$\to \quad \frac{dr}{d\varphi} = \frac{r^2}{\lambda} \sqrt{\varepsilon + 2\frac{\alpha}{r} - \frac{\lambda^2}{r^2}}$$

$$\boxed{\frac{dr}{d\varphi} = r^2 \left(\frac{\varepsilon}{\lambda^2} + 2\frac{\alpha}{\lambda^2}\frac{1}{r} - \frac{1}{r^2} \right)^{1/2}}$$

$$\hookrightarrow \quad r(\varphi) \; !$$

Trennung der Variablen:

$$\int_{r_0}^{r_\varphi} \frac{dr}{r^2 \sqrt{A + 2B\frac{1}{r} - \frac{1}{r^2}}} = \varphi - \varphi_0$$

$$A = \frac{\varepsilon}{\lambda^2}$$

$$B = \alpha / \lambda^2$$

Subst.: $\quad r = \frac{1}{u} \; ; \quad dr = -\frac{1}{u^2} du$

$$\underline{\underline{\varphi - \varphi_0}} = \int_{1/r_\varphi}^{1/r_0} \frac{du}{\sqrt{\underbrace{A + 2Bu - u^2}_{= A + B^2 - (u-B)^2}}}$$

$$= \int_{1/r_\varphi - B}^{1/r_0 - B} \frac{dx}{\sqrt{\underbrace{A + B^2}_{v^2} - x^2}}$$

$$u = x + B$$

$$= \underline{\underline{\arccos \frac{1/r_\varphi - B}{\sqrt{A + B^2}}}} - \underbrace{\dots}_{= \varphi_0 \,!}$$

$$\arccos' \frac{x}{v} = \frac{-1}{\sqrt{v^2 - x^2}}$$

$$\rightsquigarrow \quad \cos\varphi = \frac{{}^{1}/_{r_\varphi} - B}{\sqrt{A + B^2}}$$

$$\rightsquigarrow \quad \boxed{r(\varphi) = \frac{1/B}{1 + \sqrt{1 + \frac{A}{B^2}}\,\cos\varphi}}$$

$$\equiv \frac{p}{1 + \varepsilon\cos\varphi}$$

mit $\quad p = {}^{1}/_{B} = {}^{\lambda^2}/_{\alpha}$

$$\varepsilon = \left(1 + \frac{A}{B^2}\right)^{1/2} < 1$$

$\rightsquigarrow$ Planetenbahn $\hat{=}$ Ellipse!