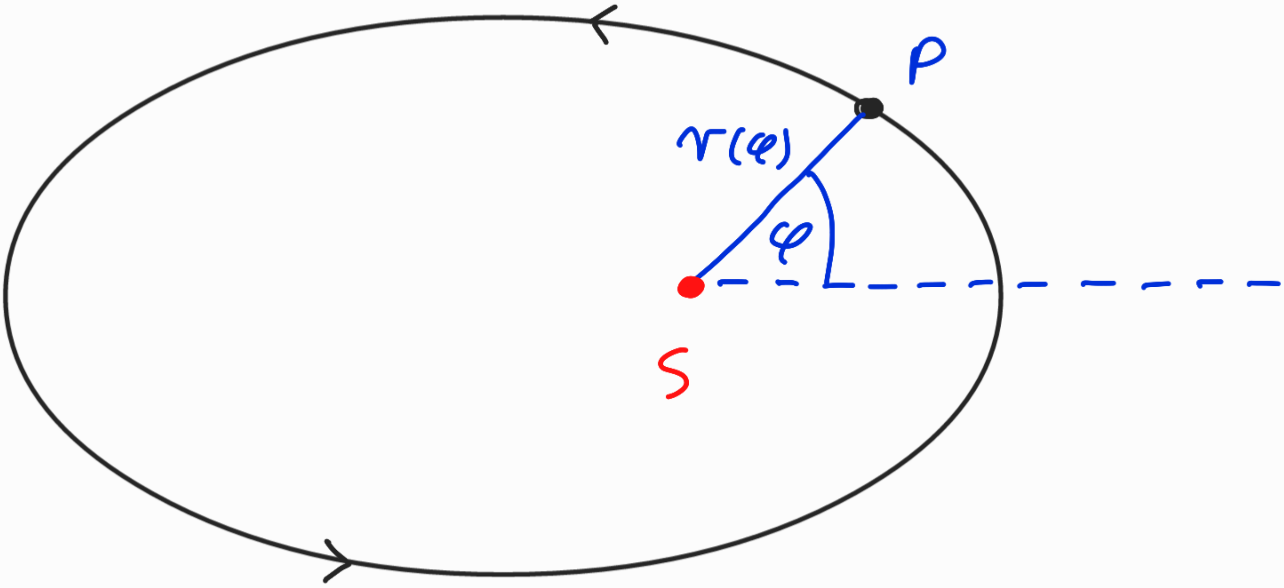
letzte Vrlsg.:

- ## Planetenbahnen:



$$r(\varphi) = \,?$$

Radialgleichung:

$$\boxed{E = \frac{m}{2}\dot{r}^2 - \frac{\alpha m}{r} + \lambda^2 \frac{m}{r^2}}$$

$$U(r) = -\frac{\alpha m}{r} \quad ; \quad \alpha = GM$$

$$\lambda = \ell/m$$

$$\rightarrow \quad \boxed{\dot{r}^2 - 2\frac{\alpha}{r} + \frac{\lambda^2}{r^2} = \varepsilon} := \frac{2E}{m}$$

$$\rightarrow \quad \frac{dr}{dt} = \sqrt{\varepsilon + \frac{2\alpha}{r} - \frac{\lambda^2}{r^2}}$$

mittels $\dot{\varphi} = \ell / m r^2 = \lambda / r^2$:

$$\boxed{\frac{dr}{d\varphi} = r^2 \sqrt{\frac{\varepsilon}{\lambda^2} + \frac{2\alpha}{\lambda^2}\frac{1}{r} - \frac{1}{r^2}}}$$

DGL 1. Ord. für $r(\varphi)$!

Trennung der Variablen:

$$\int_{r_0}^{r_\varphi} \frac{dr}{r^2 \left(A + \frac{2B}{r} + \frac{1}{r^2}\right)^{1/2}} = \varphi - \varphi_0$$

$A = \varepsilon / \lambda^2$

$B = \alpha / \lambda^2$

$$\to \quad \boxed{r(\varphi) = \frac{p}{1 + \varepsilon \cos\varphi}} \qquad (*)$$

mit $\quad p = \dfrac{\lambda^2}{\alpha}$

$$\varepsilon = \left(1 + \frac{e\,\lambda^2}{\alpha^2}\right)^{1/2}$$

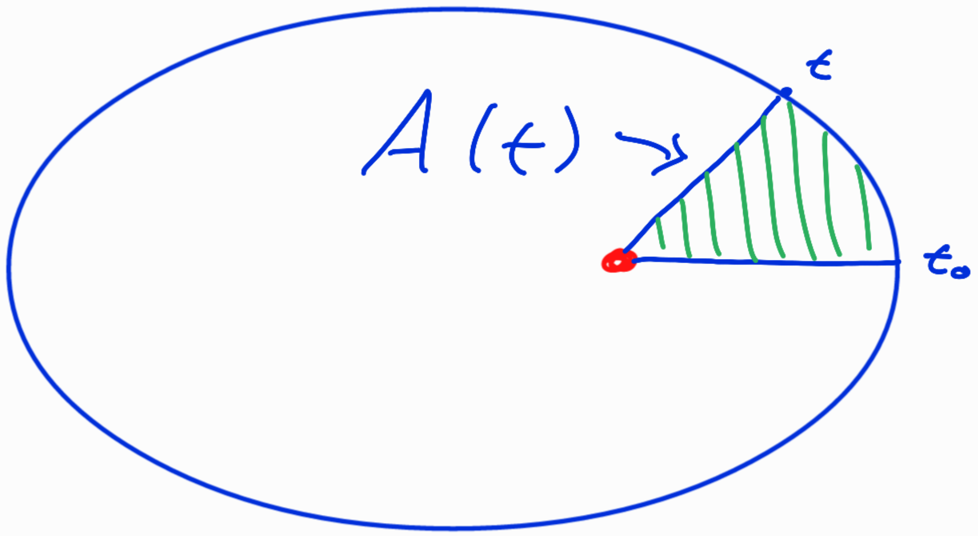
- $E < 0 \quad \to \quad e = \dfrac{2E}{m} < 0$

$$\to \quad \underline{\varepsilon < 1}$$

d.h. Planet bewegt sich auf <u>Ellipse</u> (*), Sonne im Brennpunkt.

(<u>1. Keplersche Gesetz</u>)

2. Keplersche Gesetz = Flächensatz:



$$\boxed{A(t) = w_0 t}$$

$$w_0 = \frac{l}{2m} = \frac{\lambda}{2} = \text{konst.}$$

3. Keplersche Gesetz:

für alle Planeten ist

$$\boxed{\frac{T^2}{a^3} = \text{konst.}}$$

T: Umlaufzeit

a: große Halbachse

z. z.: $T = u \, a^{3/2}$

$\uparrow$ universelle Konst.!

$$T \overset{!}{=} \frac{A_{\text{Ellipse}}}{w_0} = \frac{\pi a b}{\lambda/2}$$

$\uparrow$ Flächensatz

$$= \underbrace{\frac{2\pi b}{\lambda\sqrt{a}}} \cdot a^{3/2}$$

$\uparrow$ konstant!

$$\frac{b}{\lambda\sqrt{a}} = \frac{1}{\lambda}\sqrt{\underbrace{(1-\varepsilon^2)a}_{\llcorner p}} = \frac{\sqrt{p}}{\lambda} = \frac{1}{\sqrt{\alpha}}!$$

$b = \sqrt{1-\varepsilon^2}\, a$

$p = \frac{\lambda^2}{\alpha}$

Übg.: Runge-Lenz-Vektor:

$$\vec{A} := \vec{p}(t) \times \vec{L} - \gamma m \frac{\vec{r}(t)}{r(t)}$$

$$U(r) = -\gamma/r$$

konstant!

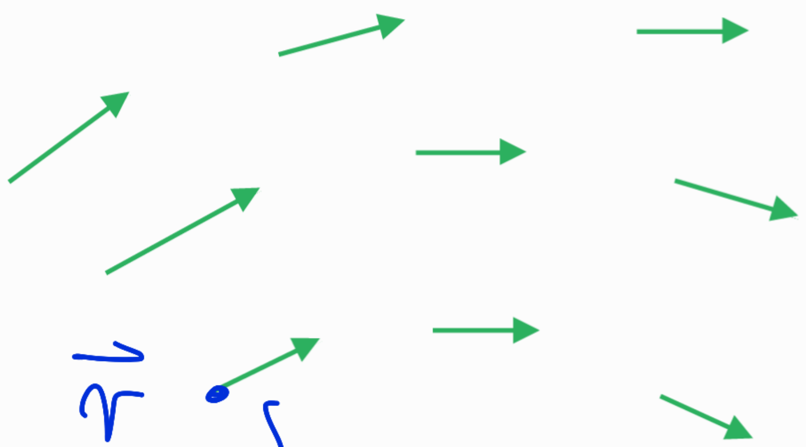
Vektoranalysis

insb. für Elektrostatik, Magnetostatik, Elektrodynamik } Exp.ph. 2

Divergenz eines Vektorfelds

$\Rightarrow$ Satz von Gauß

allg. Vektorfeld $\vec{A}$



$\vec{r}$

$\vec{A}(\vec{r})$

$$\vec{A} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\vec{r} \longmapsto \vec{A}(\vec{r})$$

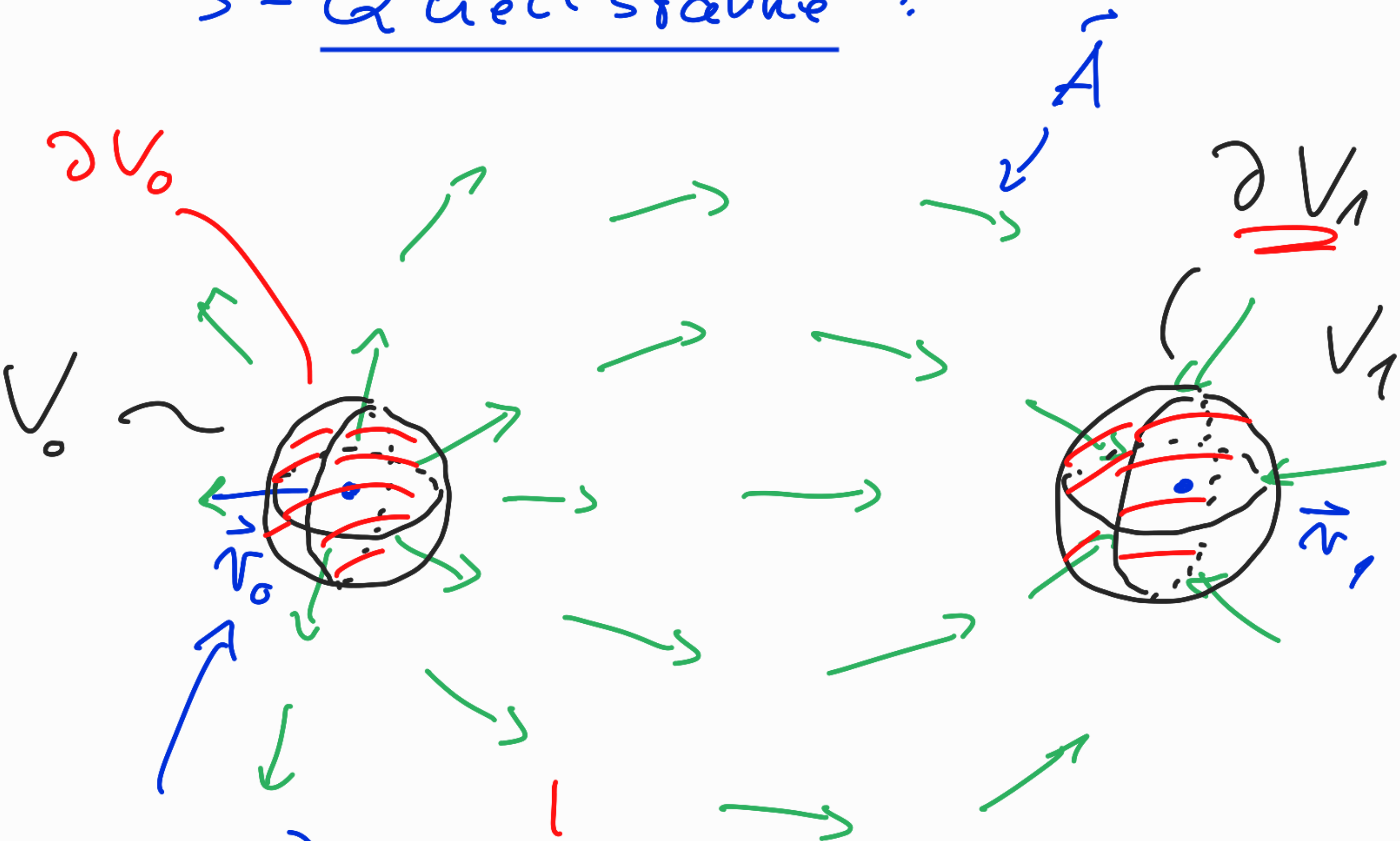
Magnetfeld: $\vec{F}(\vec{r})$

Strömungsfeld: $\vec{j}(\vec{r})$

Divergenz eines Vektorfeldes $\vec{A}$

2) $\operatorname{div} \vec{A} = \,?$

$= $ Quellstärke :



$\operatorname{div} \vec{A}(\vec{r}_0) \overset{!}{>} 0$ $\qquad$ $\operatorname{div} \vec{A}(\vec{r}_1) \overset{!}{<} 0$

Quelle ! $\qquad$ Senke !

$\oint_{\partial V_0} \vec{A}\, d\vec{f} \overset{!}{>} 0$ $\qquad$ $\oint_{\partial V_1} \vec{A}\, d\vec{f} \overset{!}{<} 0$

Def.:

Divergenz (Quellstärke) von VF. $\vec{A}$ in $\vec{r}_0$:

$$\boxed{\operatorname{div} \vec{A}(\vec{r}_0) := \lim_{|V| \to 0} \frac{1}{|V|} \int_{\partial V} \vec{A} \, d\vec{f}}$$

V: Volumen um r_0

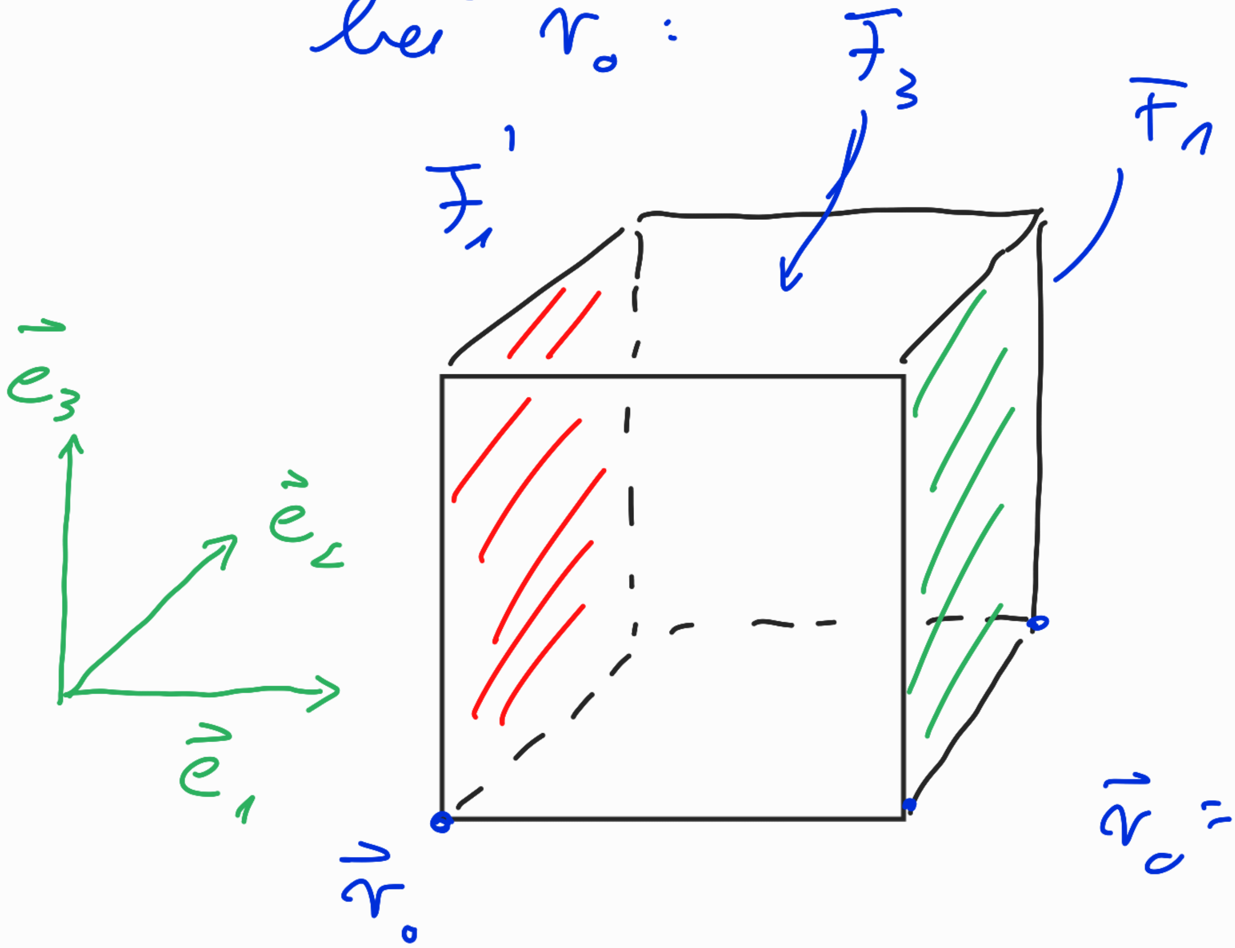
∂V: Rand (Oberfläche) von V

$|V|$: Volumeninhalt von V

Berechnung der Divergenz in kartesischen Koordinaten:

V : Würfel mit Kantenlänge $\underline{\underline{h \to 0}}$

bei $\vec{r}_0$:



$$\vec{r}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

$$\partial V = F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3'$$

$$\int_{\partial V} \vec{A}\, d\vec{f} = \underbrace{\int_{F_1 \cup F_1'} \vec{A}\, d\vec{f}}_{\mathrm{I}} + \underbrace{\int_{F_2 \cup F_2'} \vec{A}\, d\vec{f}}_{\mathrm{II}} + \underbrace{\int_{F_3 \cup F_3'} \vec{A}\, d\vec{f}}_{\mathrm{III}}$$

$$\mathrm{I} = \int_{y_0}^{y_0+h} \int_{z_0}^{z_0+h} \underbrace{\left(A_1(x_0+h, y, z) - A_1(x_0, y, z)\right)}_{\simeq \frac{\partial A_1}{\partial x}\big|_{x_0, y_0, z_0} \cdot h} \underbrace{dy\, dz}_{h^2}$$

$$\int_{\partial V} \vec{A}\, d\vec{f} = \left(\frac{\partial A_1(\vec{r}_0)}{\partial x} + \frac{\partial A_2(\vec{r}_0)}{\partial y} + \frac{\partial A_3(\vec{r}_0)}{\partial z} \right)$$

$$h^3 = |V| \qquad \vec{r}_0 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\rightarrow \boxed{\operatorname{div} \vec{A}(\vec{r}_0) = \sum_{i=1}^{3} \frac{\partial A_i(\vec{r}_0)}{\partial x_i}}$$

mittels $\vec{\nabla} = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix}$:

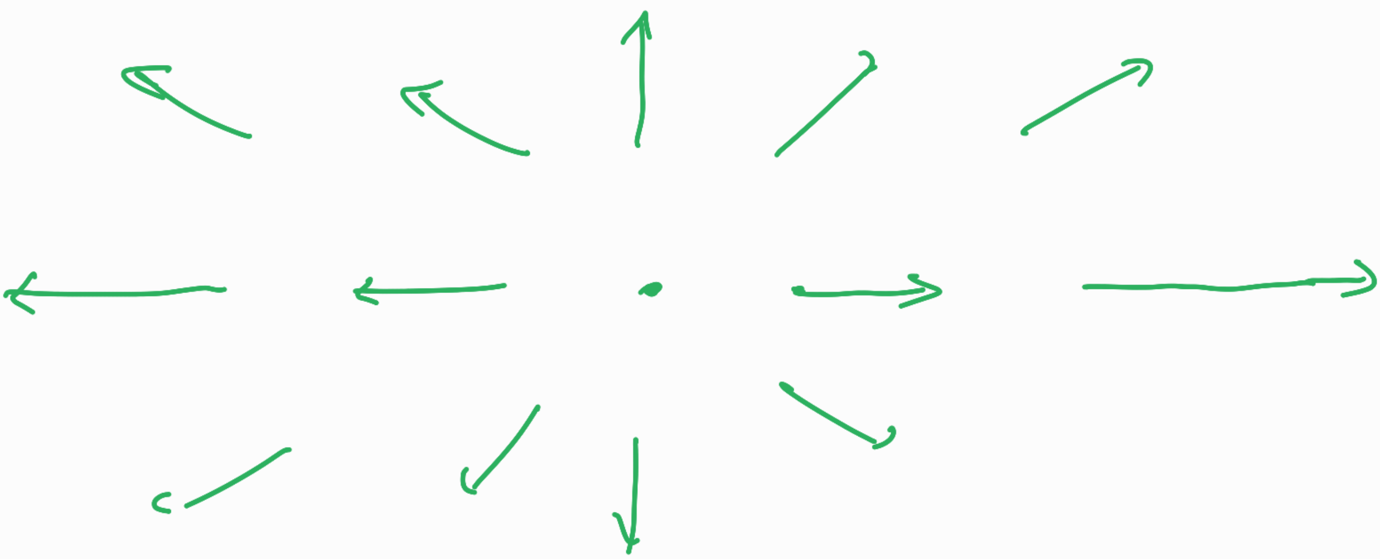
$$\boxed{\operatorname{div} \vec{A}(\vec{r}_0) = \vec{\nabla} \cdot \vec{A}}$$

$$\vec\nabla\cdot\vec A = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \cdot \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

$$= \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

<u>Beispiele</u>:

1) Vf $\vec A : \vec r \mapsto \vec r$

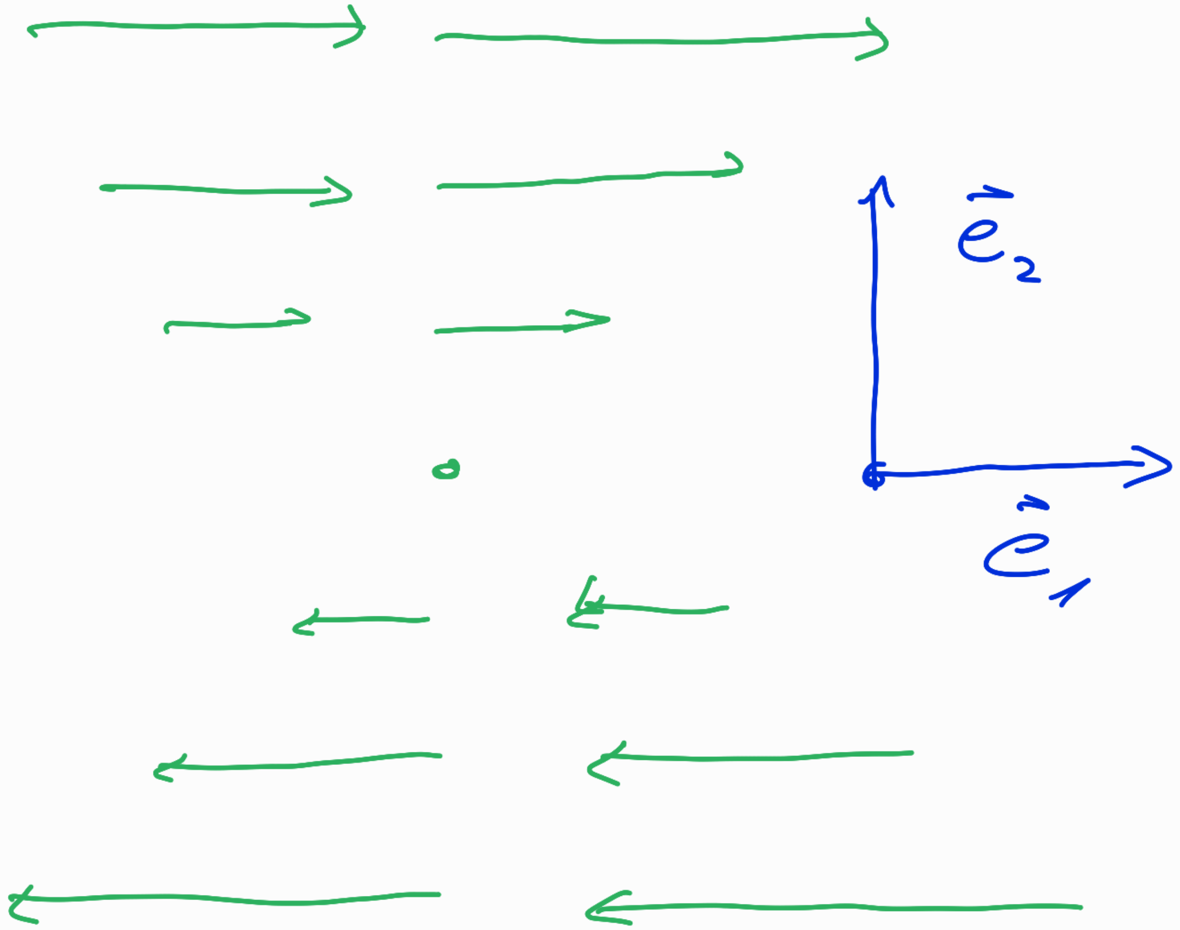


$$\vec{A} : \vec{r} \mapsto \vec{r} \qquad \vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\operatorname{div} \vec{A} = \operatorname{div} \vec{r}$$

$$\boxed{\operatorname{div} \vec{r} = \sum_{i=1}^{3} \underbrace{\frac{\partial x_i}{\partial x_i}}_{=1} = 3}$$

2) $$\vec{A} : \vec{r} \mapsto x_2 \vec{e}_1 = \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix}$$

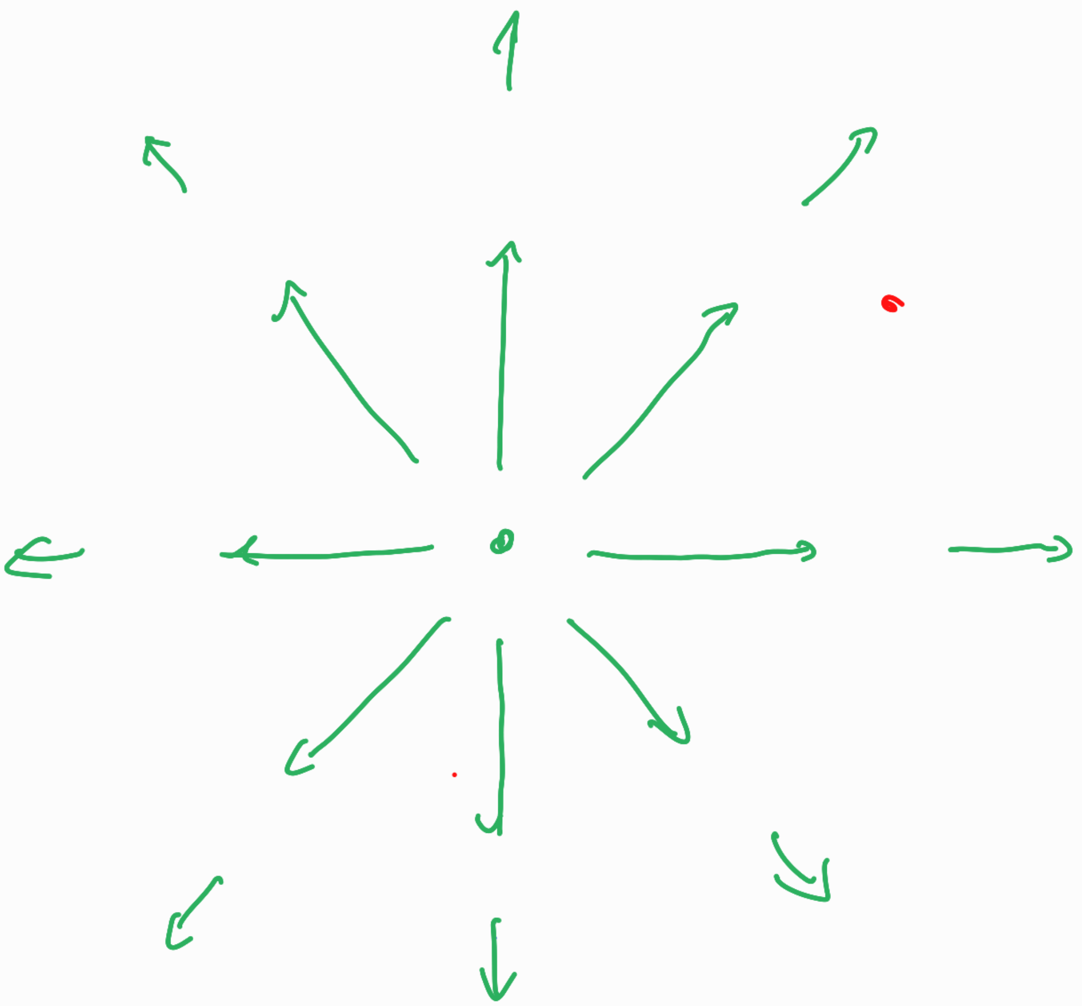


$$\operatorname{div}\begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix} = \overset{c}{\cancel{\frac{\partial x_2}{\partial x_1}}} + \overset{c}{\cancel{\frac{\partial 0}{\partial x_2}}} + \overset{c}{\cancel{\frac{\partial 0}{\partial x_3}}}$$

$$= 0 \; !$$

3) Vf: $\vec{A} : \vec{r} \longmapsto \frac{\hat{r}}{r^2}$

$(\vec{r} \neq \vec{0})$



$$\boxed{\operatorname{div} \frac{\hat{r}}{r^2} \overset{!}{=} 0}$$

$$\operatorname{div} \frac{\hat{r}}{r^2} = \operatorname{div} \frac{\vec{r}}{r^3} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{x_i}{r^3}\right)$$

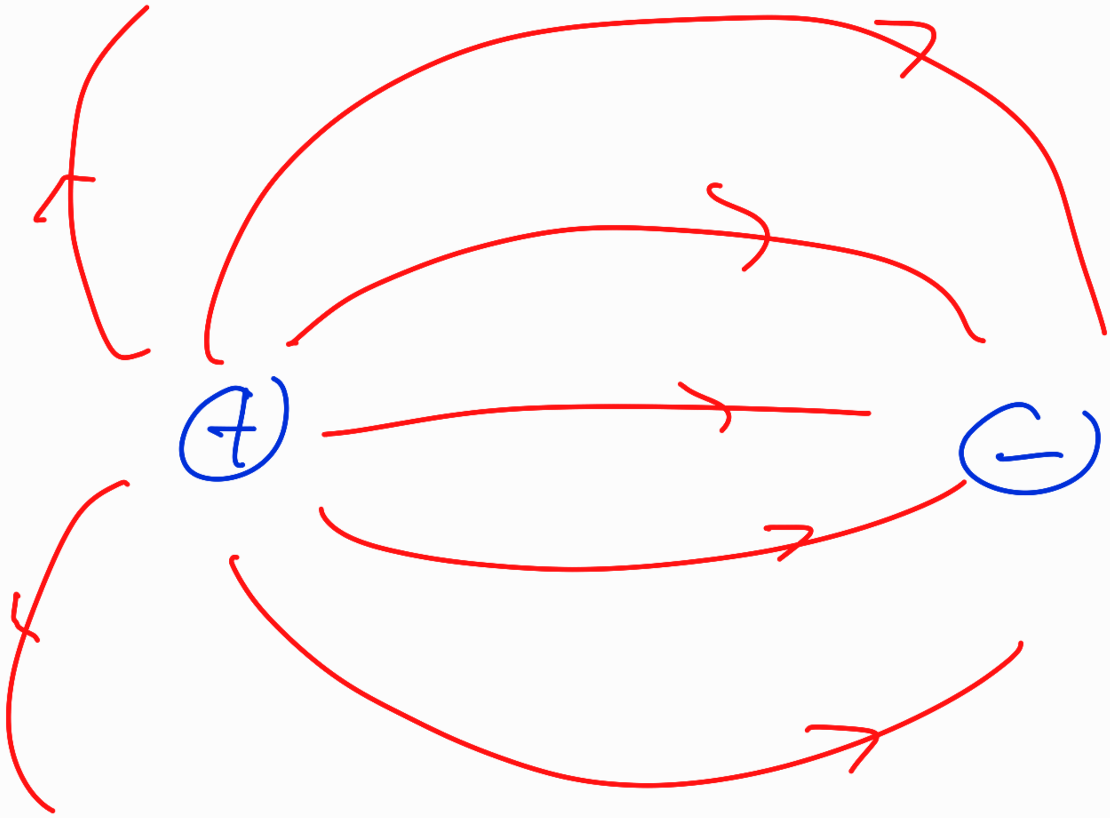
$$= \sum_{i=1}^{3}\left(\frac{1}{r^3} - \frac{3}{\cancel{2}} \frac{x_i}{r^5} \cancel{2} x_i\right)$$

$$= \sum_{i=1}^{3}\left(\frac{1}{r^3} - 3\frac{x_i^2}{r^5}\right)$$

$$= \frac{3}{r^3} - \frac{3}{r^{\cancel{5}\,3}} \cancel{r^2} = 0 \; !$$

Anwendung in Elektrostatik:

Gaußsche Gesetz:



„elektrische Ladungen sind Quellen / Senken des elektr. Felds"

$$\frac{\underline{\text{elektr. Ladg.}}}{\text{Volumen}} = \rho(\vec{r}) : \underline{\text{Ladgs.}}\ \underline{\text{dichte}}$$

<u>elektr. Feld</u> : $\vec{E}(\vec{r})$

<u>Gaußsche Gesetz</u> :

$$\boxed{\operatorname{div} \vec{E}(\vec{r}) \overset{!}{=} \frac{1}{\varepsilon_0}\, \rho(\vec{r})}$$

$$\varepsilon_0 = \ldots$$