Letzte Vrlsg.:

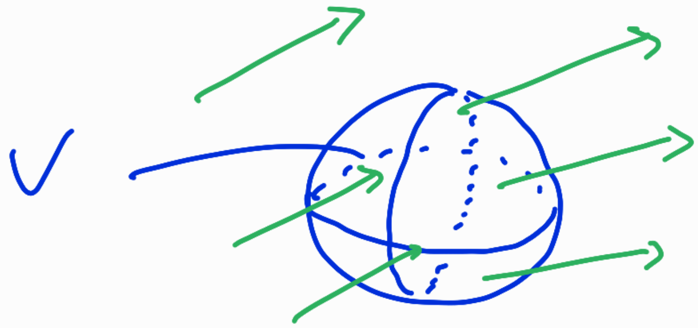
- **Divergenz** (Quellstärke) eines Vektorfelds:

$$\boxed{\operatorname{div} \vec{A}(\vec{r}) = \lim_{|V| \to 0} \frac{1}{|V|} \int_{\partial V} \vec{A}\, d\vec{f}}$$

V: Volumen um $\vec{r}$

∂V: Rand (=Oberfläche) von V

$|V|$: Volumeninhalt von V



$$I_{\partial V} = \int_{\partial V} \vec{A}\, d\vec{f}$$

- Berechnung in kartesischen Koord.:

$$\vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} , \quad \vec{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

$$\rightarrow \boxed{\begin{aligned} \operatorname{div} \vec{A} &= \sum_{i=1}^{3} \frac{\partial A_i}{\partial x_i} \\ &= \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} \\ &= \vec{\nabla} \cdot \vec{A} \end{aligned}}$$

Beispiele:

$$\operatorname{div} \vec{r} = 3$$

$$\operatorname{div}(x_2 \vec{e}_1) = 0$$

$$\operatorname{div} \frac{\hat{r}}{r^2} = 0 \qquad (\vec{r} \neq 0)$$

Vektorfeld $\xrightarrow{\text{div}}$ Skalarfeld

$\vec{A}$ $\qquad\qquad$ $\text{div}\,\vec{A}$

- Anwendung in Elektrostatik:

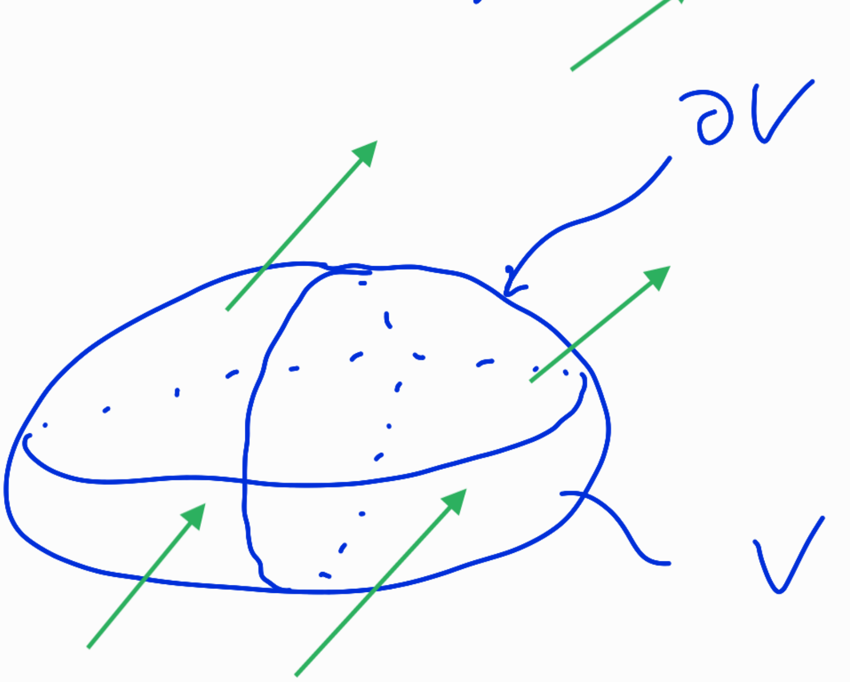
Gaußsches Gesetz:

$$\boxed{\text{div}\,\vec{E} \overset{!}{=} \frac{1}{\varepsilon_0}\,\varrho}$$

$\left(\varepsilon_0 = 8{,}85 \cdot 10^{-12}\,\frac{As}{Vm}\right)$

„Elektrische Ladungen sind Quellen des elektr. Felds."

Satz von Gauß



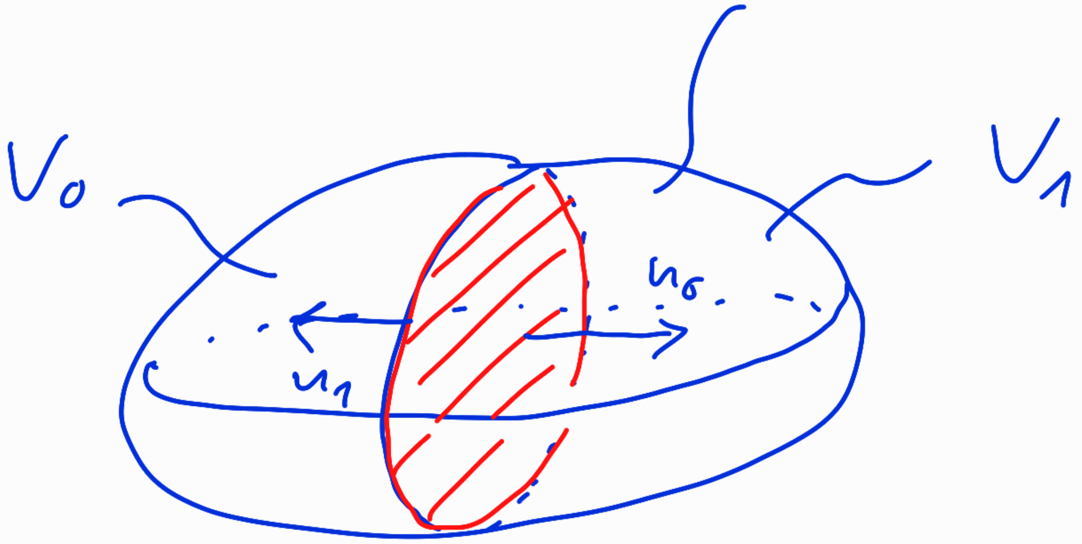
$$\boxed{\int_{\partial V} \vec{A}\, d\vec{f} \overset{!}{=} \int_V \operatorname{div} \vec{A}\, dV}$$

Netto-Fluss von $\vec{A}$ durch ∂V

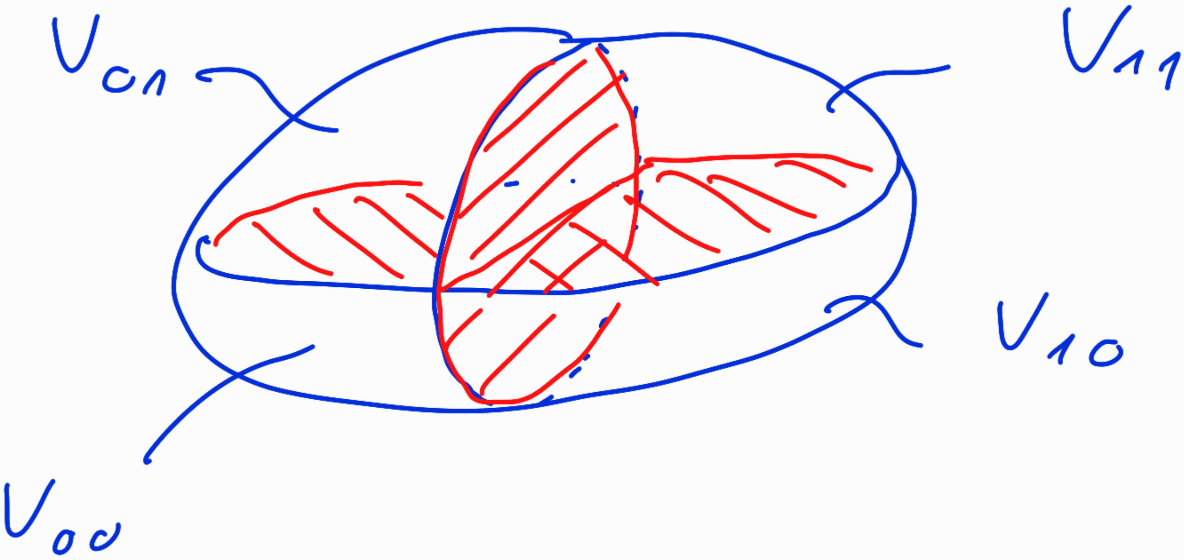
Gesamtquellstärke in V

1. „Physikbeweis"

$V = V_0 \cup V_1$

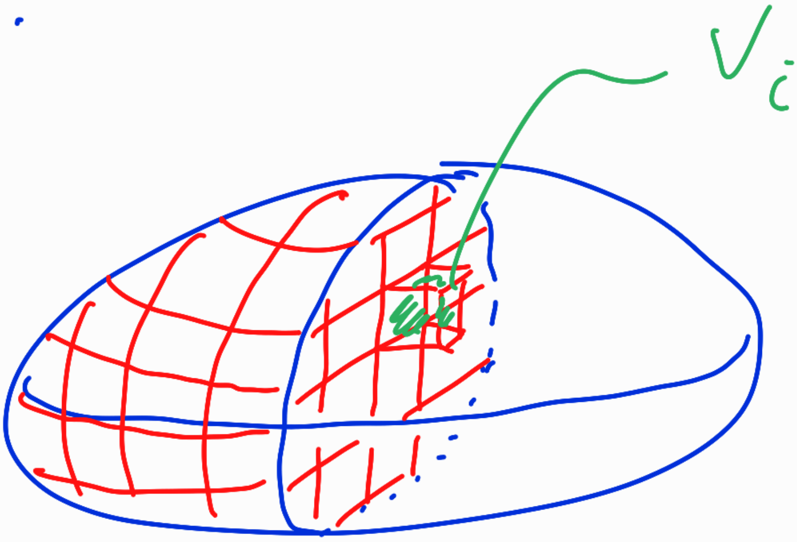


$$\int_{\partial V} \vec{A}\, d\vec{f} \overset{!}{=} \int_{\partial V_0} \vec{A}\, d\vec{f} + \int_{\partial V_1} \vec{A}\, d\vec{f}$$



$$\int_{\partial V} \vec{A}\, d\vec{f} \overset{!}{=} \int_{\partial V_{00}} \vec{A}\, d\vec{f} + \int_{\partial V_{01}} \vec{A}\, d\vec{f} + \int_{\partial V_{10}} \vec{A}\, d\vec{f} + \int_{\partial V_{11}} \vec{A}\, d\vec{f}$$

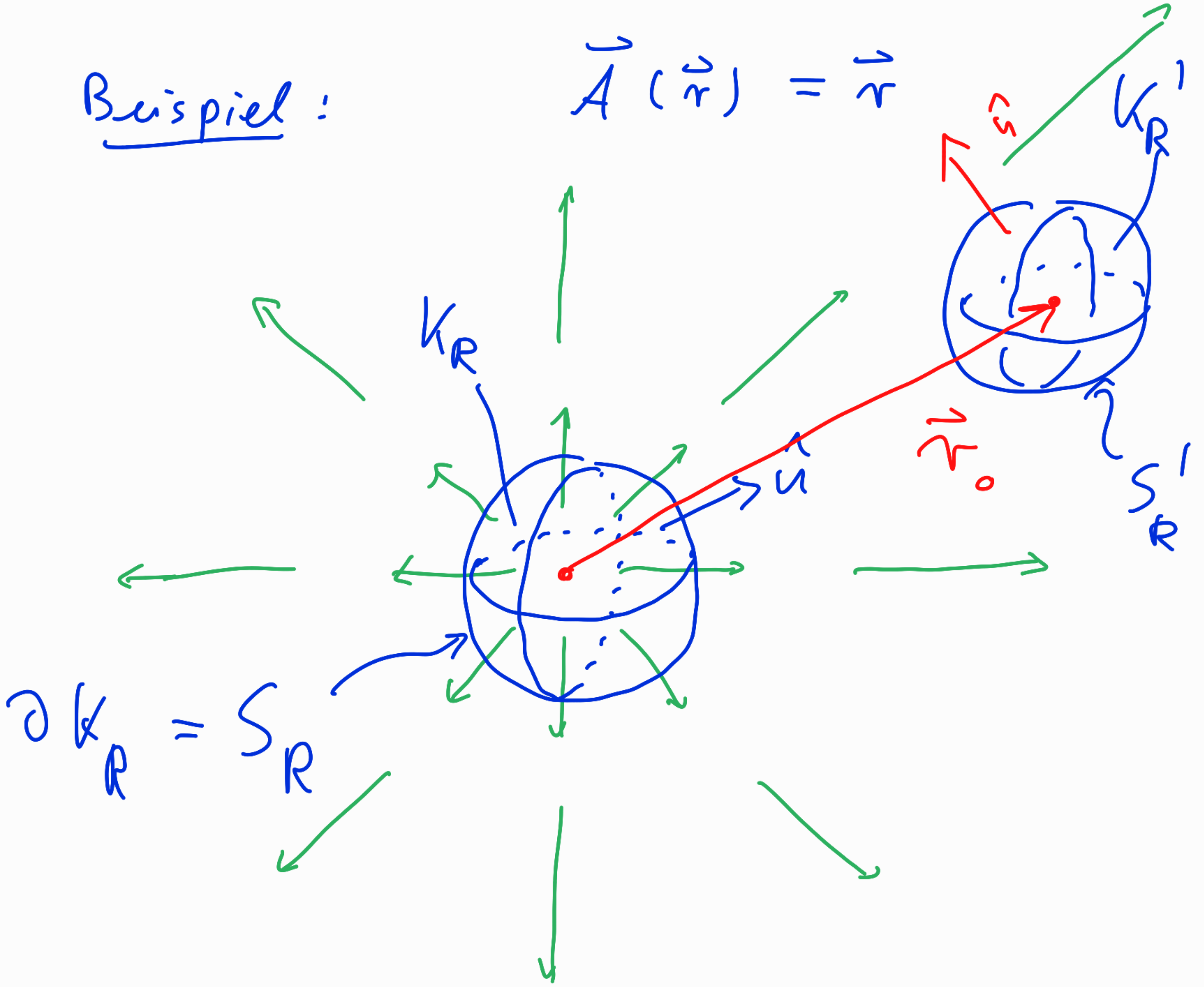
$\vdots$



$$\int_{\partial V} \vec{A}\, d\vec{f} = \underbrace{\sum_{i=1}^{N}}_{\int} \underbrace{\frac{1}{|V_i|} \int_{\partial V_i} \vec{A}\, d\vec{f}}_{= \operatorname{div} \vec{A}(\vec{r}_i)} \underbrace{|V_i|}_{= dV}$$

$$= \int_V \operatorname{div} \vec{A}\, dV$$

Beispiel: $\vec{A}(\vec{r}) = \vec{r}$



$$\operatorname{div} \vec{A} = \operatorname{div} \vec{r} = 3$$

a)

$$\int_{S_R} \vec{A}\, d\vec{f} \overset{!}{=} 4\pi R^2 \cdot R = 4\pi R^3$$

mittels S. v. Gauß:

$$\int_{S_R} \vec{A}\, d\vec{f} = \int_{\underline{\underline{\partial K_R}}} \vec{A}\, d\vec{f} = \int_{K_R} \underbrace{\mathrm{div}\,\vec{A}}_{\overset{?}{=}3}\, dV$$

$$= 3 \int_{K_R} dV = 3 \cdot \frac{4}{3}\pi R^3 \quad \checkmark$$

b)

$$? \overset{!}{=} \int_{S_R'} \vec{A}\, d\vec{f} = \int_{\underline{\underline{\partial K_R'}}} \vec{A}\, d\vec{f}$$

S. v. Gauß

$$= \int_{K_R'} \underbrace{\mathrm{div}\,\vec{A}}_{\overset{?}{=}3}\, dV = 4\pi R^3 \; !$$

Anwendung in der Elektrostatik:

Gaußsche Gesetz

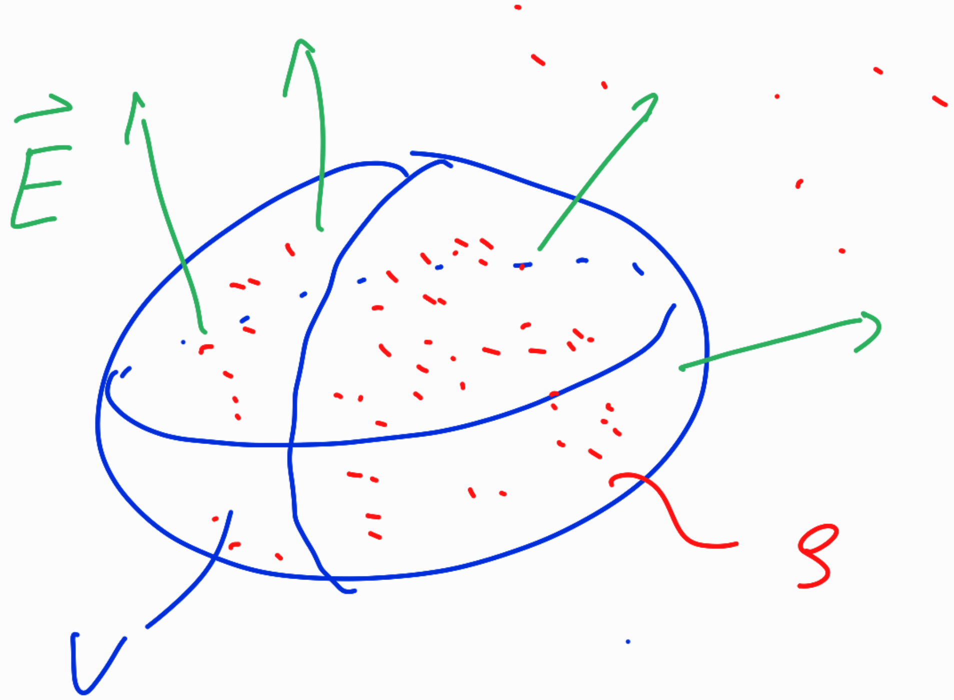
differenziell

$$\boxed{\operatorname{div} \vec{E} = \frac{1}{\varepsilon_0} \varrho}$$

S. v. Gauß

$$\boxed{\int_{\partial V} \vec{E}\, d\vec{f} \overset{!}{=} \frac{Q_V}{\varepsilon_0}}$$

integrale Form



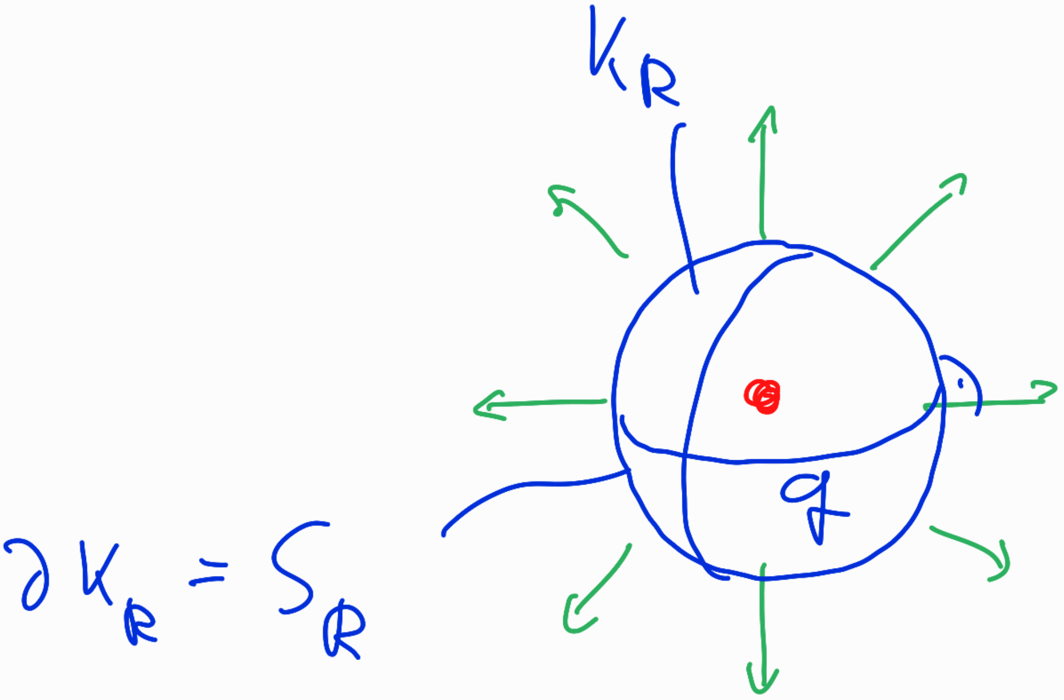
$$Q_V = \int_V \rho \, dV$$

$=$ Gesamtladung in V

$$\int_{\partial V} \vec{E}\, d\vec{f} \;=\; \text{elektr. Fluss durch } \partial V$$

$$\int_{\partial V} \vec{E}\, d\vec{f} \underset{\text{S.v.G.}}{=} \int_V \underbrace{\operatorname{div} \vec{E}}_{=\rho/\varepsilon_0} \, dV = \frac{1}{\varepsilon_0} \underbrace{\int_V \rho \, dV}_{= Q_V}$$

Beispiel: Elektrische Feld einer Punktladung q in 0?



Symmetrie: $\vec{E}(\vec{r})$ ist isotropes Zentralfeld:

$$\vec{E}(\vec{r}) = \underset{?}{\underline{\underline{E(r)}}}\,\hat{r}$$

Bestimmung der Feldstärke $E(r)$:

mittels:
$$\int_{\partial V} \vec{E}\, d\vec{f} = \frac{Q_V}{\varepsilon_0}$$

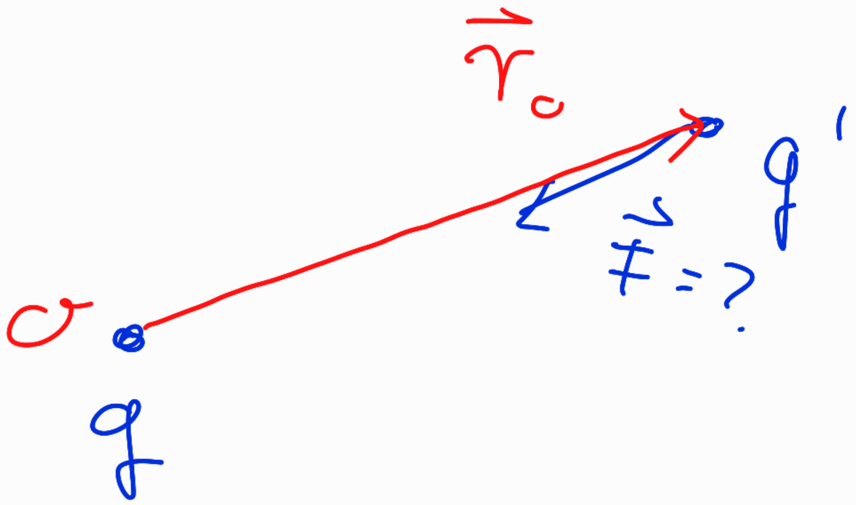
$$4\pi R^2 E(R) \overset{!}{=} \int_{S_R} \vec{E}\, d\vec{f} = \frac{1}{\varepsilon_0} q$$

$$\rightarrow \quad E(R) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R^2}$$

$$\rightarrow \quad \boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \cdot \hat{r}}$$

→

$$\rightarrow \quad \vec{F} \stackrel{!}{=} q' \vec{E}(\vec{r})$$



$$\rightarrow \quad \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

$$\vec{F} = q' \, \vec{E}(\vec{r}_0)$$

$$\boxed{\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q\,q'}{|\vec{r}_0|^2} \, \hat{r}_0}$$

Coulomb-Gesetz