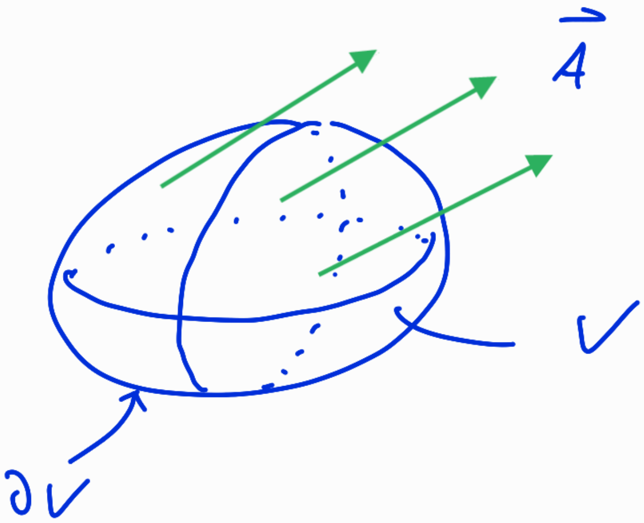
letzte Vrlsg.:

- Satz von Gauß:

$$\boxed{\int_{\partial V} \vec{A}\, d\vec{f} = \int_V \operatorname{div} \vec{A}\, dV}$$



- Anwendung in E.S.:

$$\operatorname{div} \vec{E} = \rho / \varepsilon_0$$

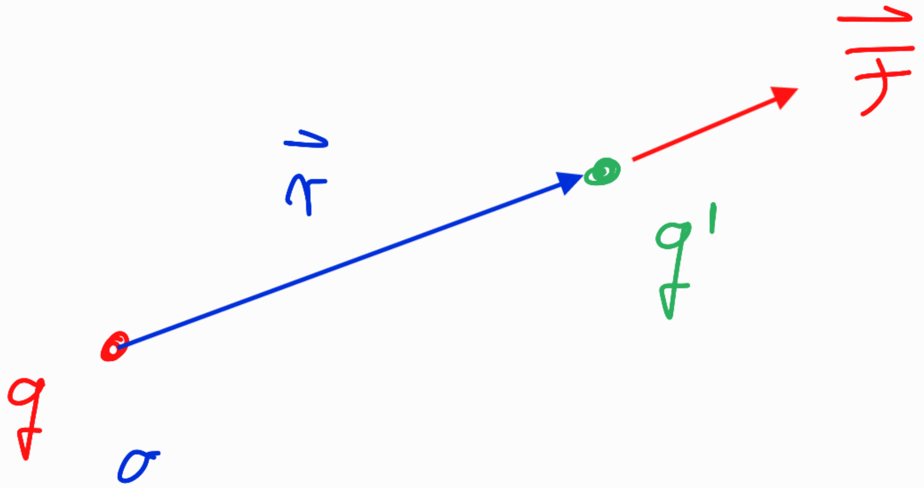
$$\hookrightarrow \int_{\partial V} \vec{E}\, d\vec{f} = \frac{Q_V}{\varepsilon_0}$$

→ Elektr. Feld einer Pkt. Ladg q in O:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

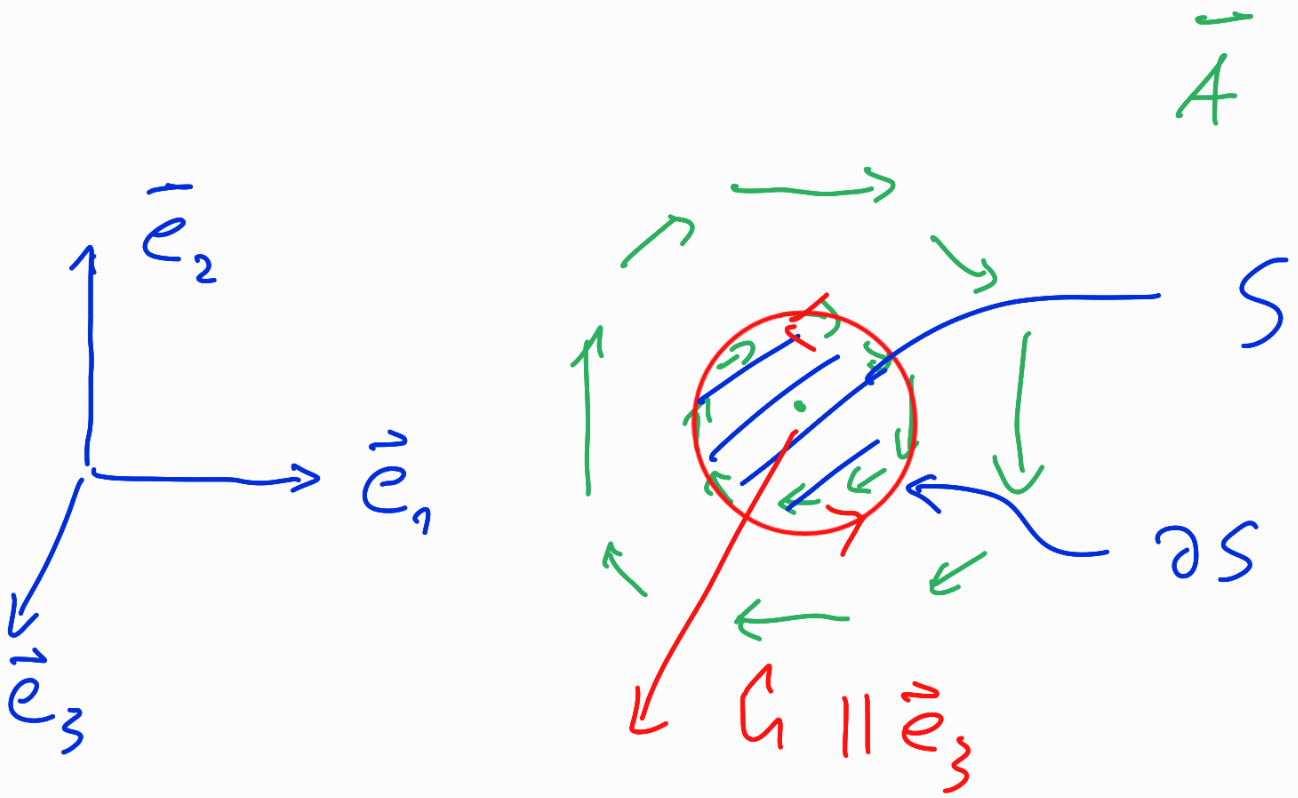
→ Kraft auf Pkt. Ladg q' in $\vec{r}$:

$$\vec{F}(\vec{r}) = q' \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q q'}{r^2} \hat{r}$$



Rotation (Wirbelstärke) eines Vf.s

$\Rightarrow$ Satz vom Stokes



$$\int_{\partial S} \vec{A}\, d\vec{\ell} \neq 0 \quad !$$

Wirbelstärke $\neq 0$!

Rotation (= Wirbelstärke) von $\vec{A}$ in $\vec{r}_0$

= Vektor $\operatorname{rot} \vec{A}(\vec{r}_0)$

eindeutig bestimmt gemäß:

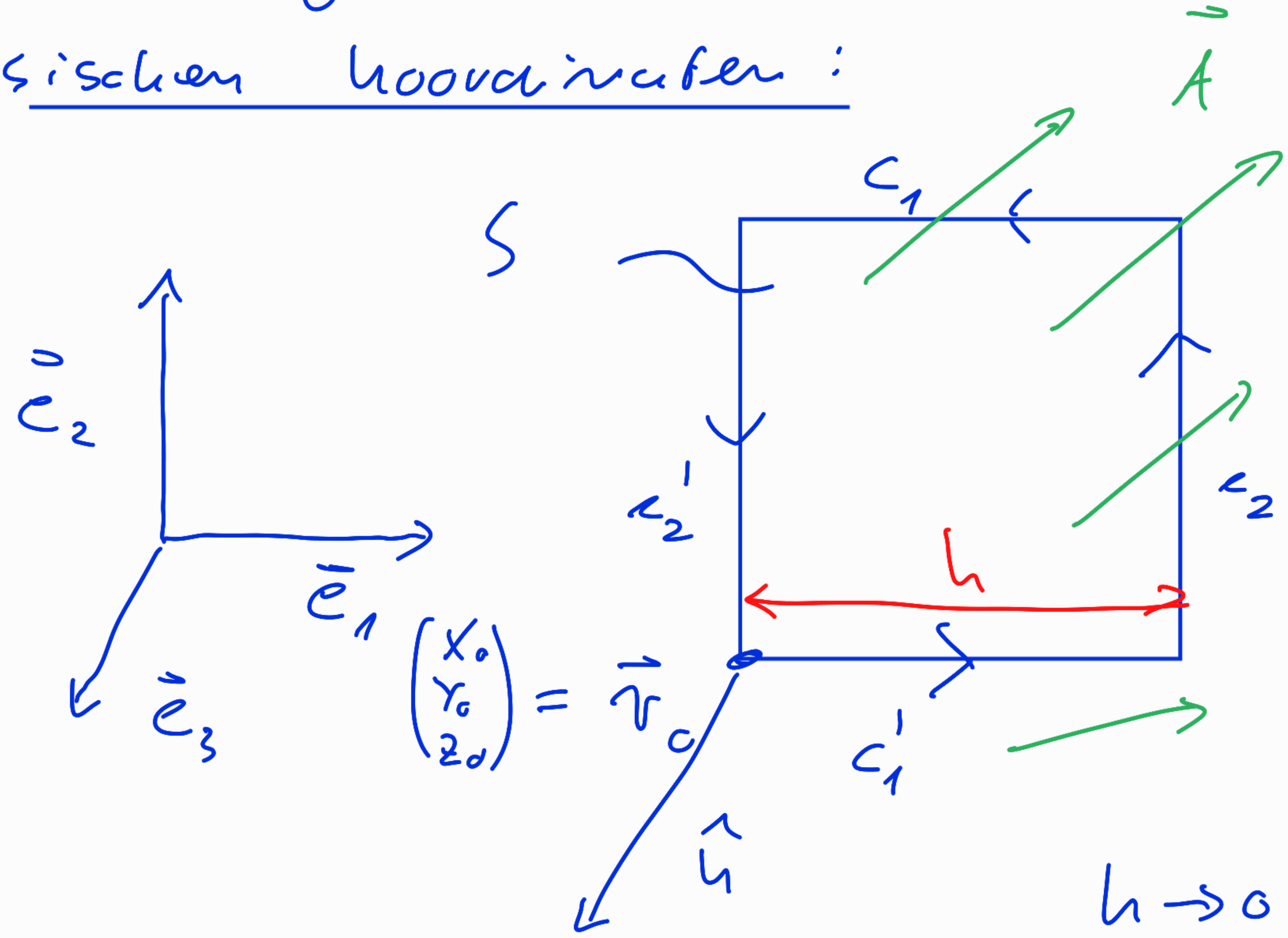
$$\boxed{\langle \hat{n}, \operatorname{rot} \vec{A}(\vec{r}_0) \rangle := \lim_{|S| \to 0} \frac{1}{|S|} \int_{\partial S} \vec{A}\, d\vec{l}}$$

S: Fläche, Flächennormale $\parallel \hat{n}$

∂S: Rand von S

$|S|$: Flächeninhalt von S

Berechnung der Rotation in kartesischen Koordinaten:



$$\partial S = c_1' + c_2 + c_1 + c_2'$$

$$\left(\operatorname{rot} \vec{A}(\vec{r}_0)\right)_3 = \langle \vec{e}_3, \operatorname{rot} \vec{A}(\vec{r}_0) \rangle$$

$$\overset{\text{Def.}}{=} \frac{1}{h^2} \int_{\partial S} \vec{A} \, d\vec{f} \quad =$$

$$= \frac{1}{h^2}\Bigg(\int_{x_0}^{x_0+h} \overbrace{A_1(x, y_0, z_0) - A_1(x, y_0+h, z_0)}^{-\frac{\partial A_1}{\partial x_2}(x, y_0, z_0)\cdot h} \, dx$$

$$+ \int_{y_0}^{y_0+h} \underbrace{A_2(x_0+h, y, z_0) - A_2(x_0, y, z_0)}_{\frac{\partial A_2}{\partial x_1}(x_0, y, z_0)\cdot h} \, dy \Bigg)$$

$$= \frac{\partial A_2}{\partial x_1}(\vec{r}_0) - \frac{\partial A_1}{\partial x_2}(\vec{r}_0)$$

$$\rightarrow \quad (\operatorname{rot} \vec{A})_1 = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}$$

$$(\operatorname{rot} \vec{A})_2 = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}$$

$$(\operatorname{rot} \vec{A})_3 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$$

d.h.:

$$\boxed{\operatorname{rot} \vec{A} = \vec{\nabla} \times \vec{A}}$$

$$= \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix} \times \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

nützliche Formeln:

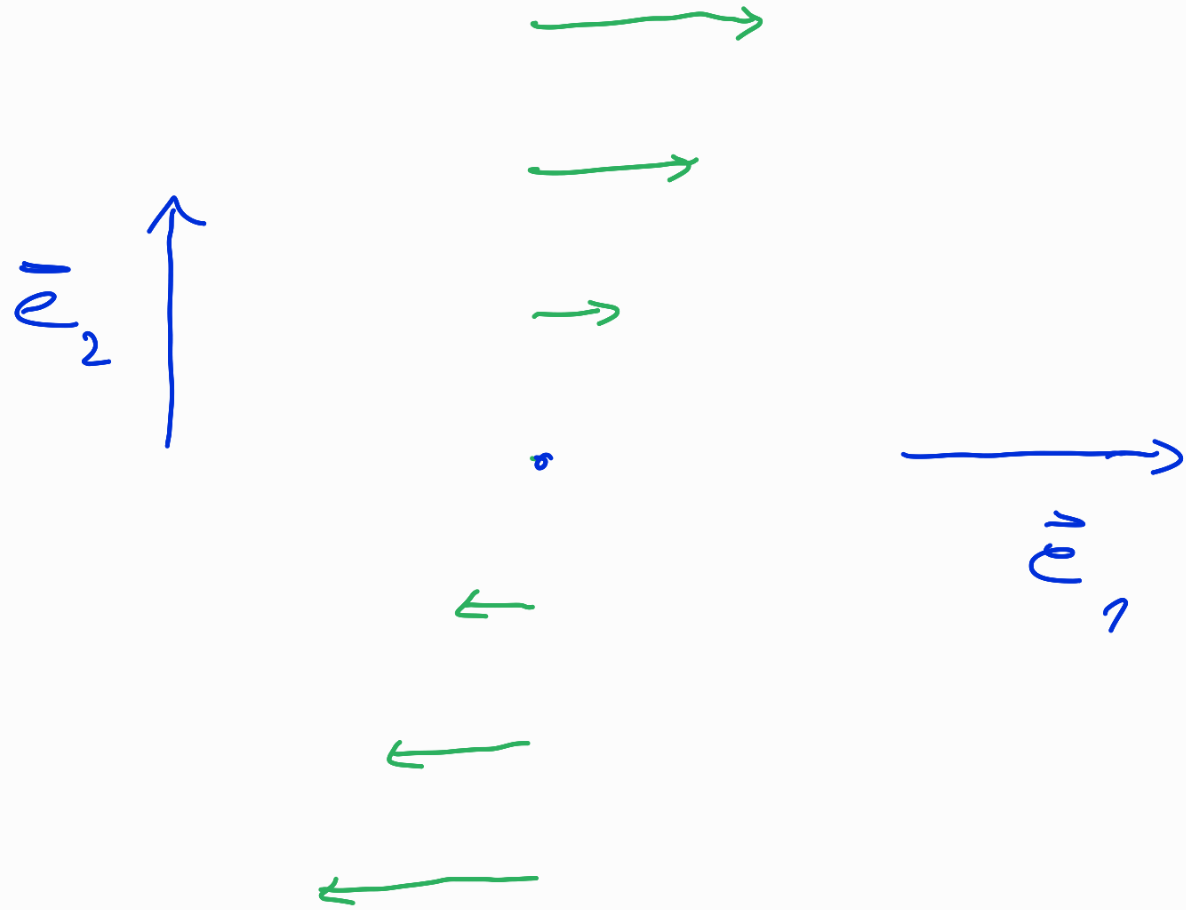
(1) $$\boxed{\operatorname{rot}(\operatorname{grad} f) = \vec{0}}$$

$\rightarrow$ konservative Vektorfelder sind rotationsfrei!

(2) $$\boxed{\operatorname{rot}(f\vec{A}) = (\operatorname{grad} f) \times \vec{A} + f \operatorname{rot} \vec{A}}$$

$\rightarrow$ Beispiele :

1)

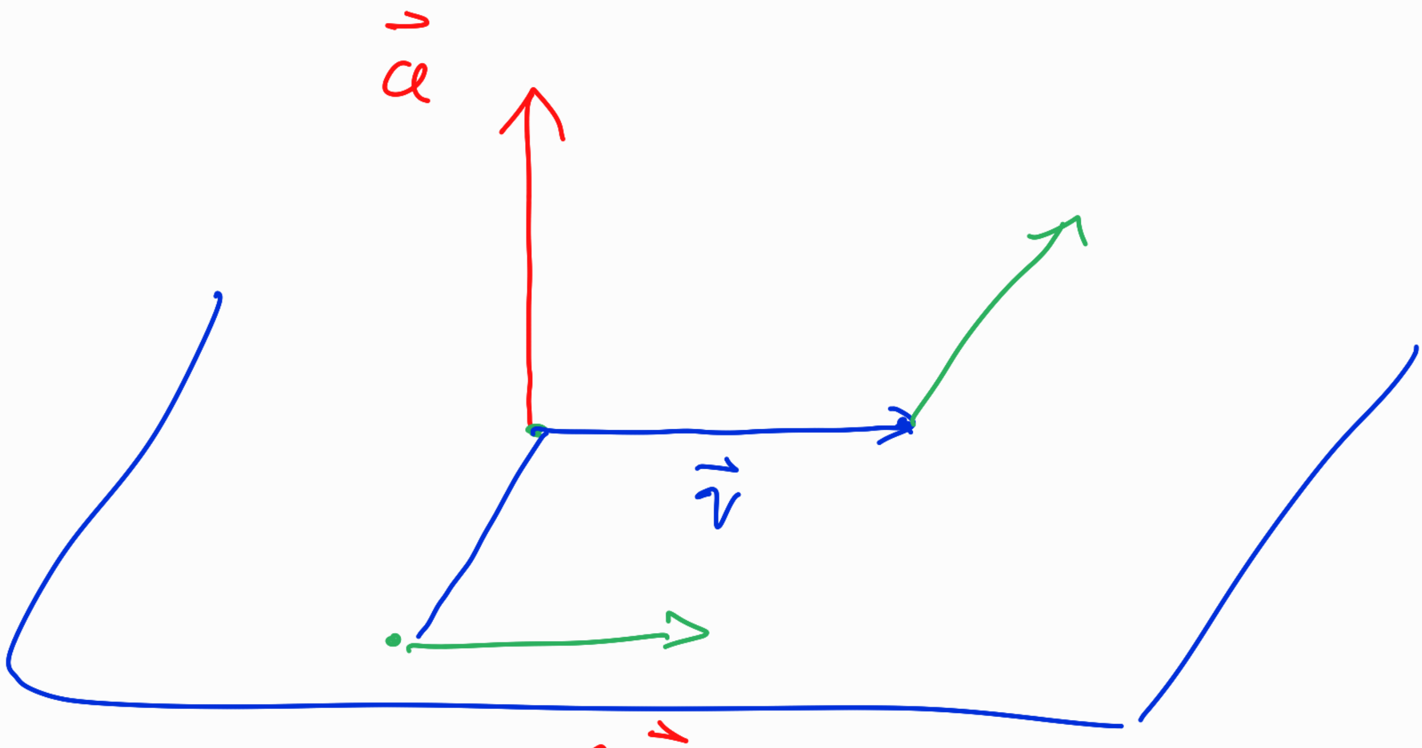


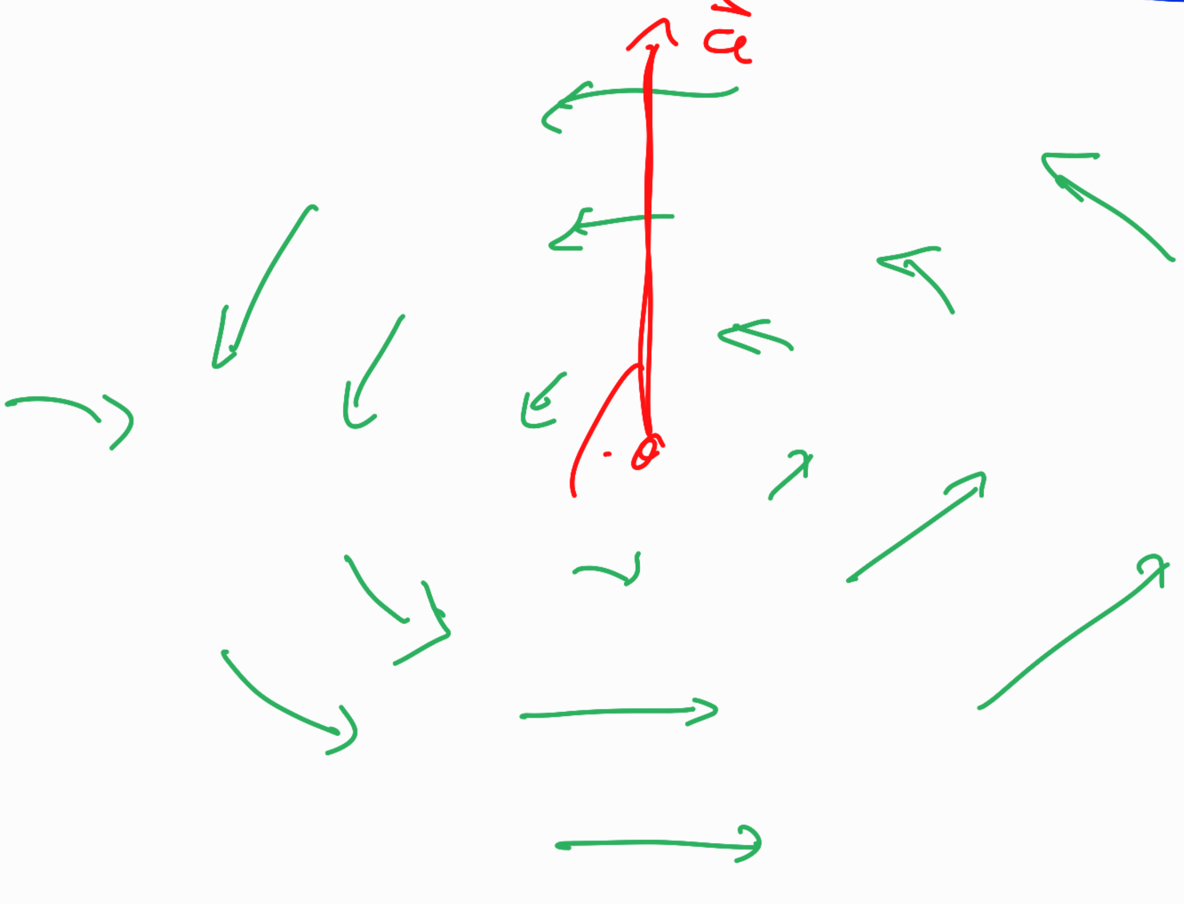
$$\vec{A}(\vec{r}) = x_2 \vec{e}_1 = \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix}$$

$$\longrightarrow \mathrm{rot}\,\vec{A} = \vec{\nabla} \times \vec{A} = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix} \times \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -\vec{e}_3$$

2) $\vec{A}(\vec{r}) = \vec{a} \times \vec{r}$





o. B. d. A. $\quad \vec{a} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}$

$$\vec{a} \times \vec{r} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} -ay \\ ax \\ 0 \end{pmatrix}$$

$$\Rightarrow \operatorname{rot}(\vec{a} \times \vec{r}) = \begin{pmatrix} 0 \\ 0 \\ 2a \end{pmatrix} = 2\vec{a}$$

$$\boxed{\operatorname{rot}(\vec{a} \times \vec{r}) = 2\vec{a}}$$

3) $\operatorname{rot} \vec{r} = \vec{\nabla} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0}$

4)
$$\vec{A}(\vec{r}) = f(r)\,\vec{r}$$

$$\operatorname{rot}\vec{A} = \operatorname{rot}(f\vec{r})$$

$$= \underbrace{\overbrace{\underbrace{(\operatorname{grad} f(r))}_{f'(r)\hat{r}} \times \vec{r}}^{=\vec{0}}} + f(r)\underbrace{\operatorname{rot}\vec{r}}_{=\vec{0}}$$

$$= \vec{0}$$

(klar, da $\vec{A}$ als isotropes Zentralfeld konservativ und damit wirbelfrei!)

Anwendungen:

1) Magnetostatik

<u>elek. Stromdichte</u> $\quad\quad$ <u>Magnetfeld</u>

$$\vec{j}(\vec{r}) \longrightarrow \vec{B}(\vec{r})$$

<u>Ampèresches Gesetz:</u>

„elektrische Ströme erzeugen magnetische Wirbelfelder":

$$\boxed{\operatorname{rot} \vec{B} \overset{!}{=} \mu_0 \vec{j}}$$

$$\left(\mu_0 = 1{,}26 \cdot 10^{6} \, \frac{Vs}{Am} \right)$$

2) <u>Faradaysche Induktions-gesetz</u>:

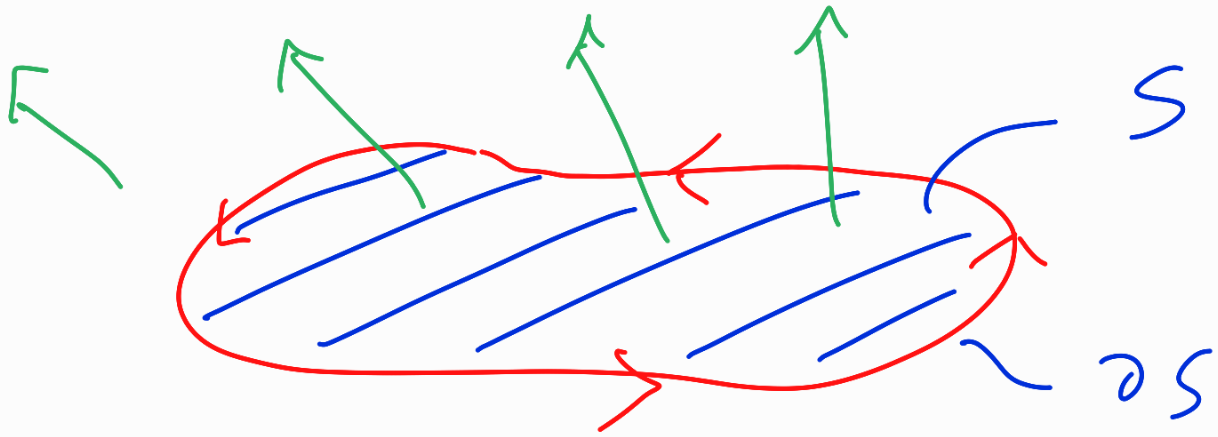
„<u>zeitlich</u> veränderliches Magnetfeld induziert elektrisches Wirbelfeld"

$$\vec{B}(\vec{r}, t) \longrightarrow \vec{E}(\vec{r}, t)$$

Faraday:

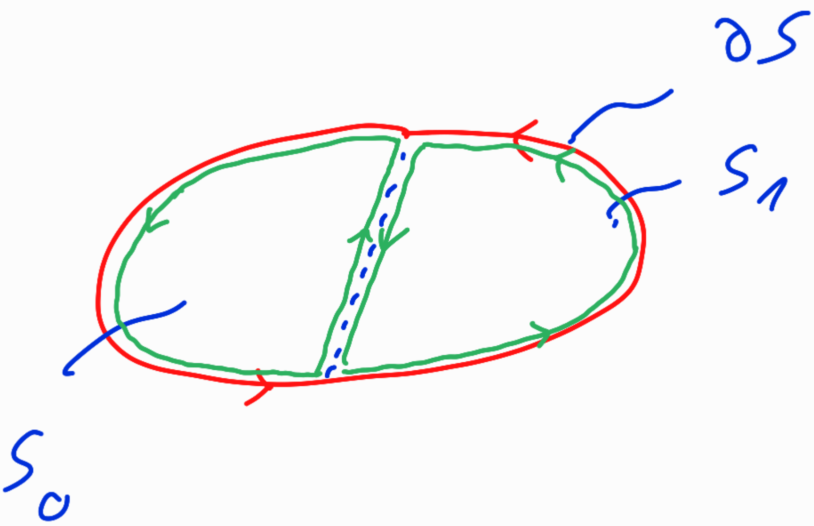
$$\boxed{\operatorname{rot}\vec{E} \overset{!}{=} -\frac{\partial \vec{B}}{\partial t}}$$

Satz vom Stokes

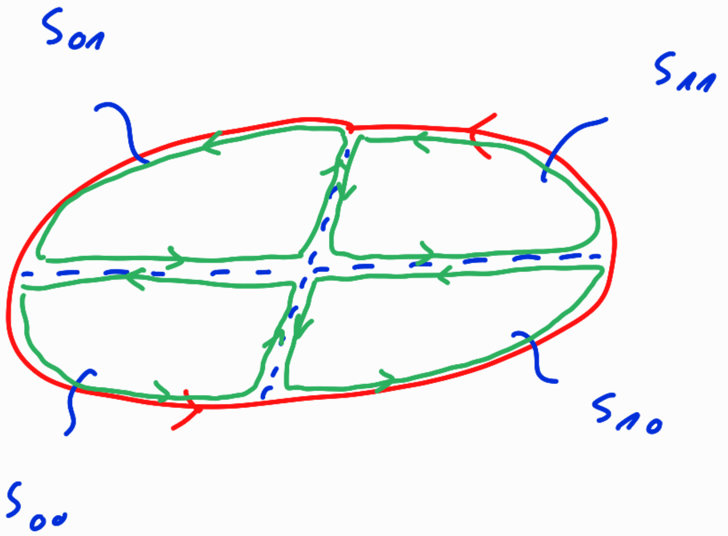


$$\boxed{\int_{\partial S} \vec{A}\, d\vec{l} \overset{!}{=} \int_{S} \operatorname{rot}\vec{A}\, d\vec{f}}$$

„Physikerbeweis":

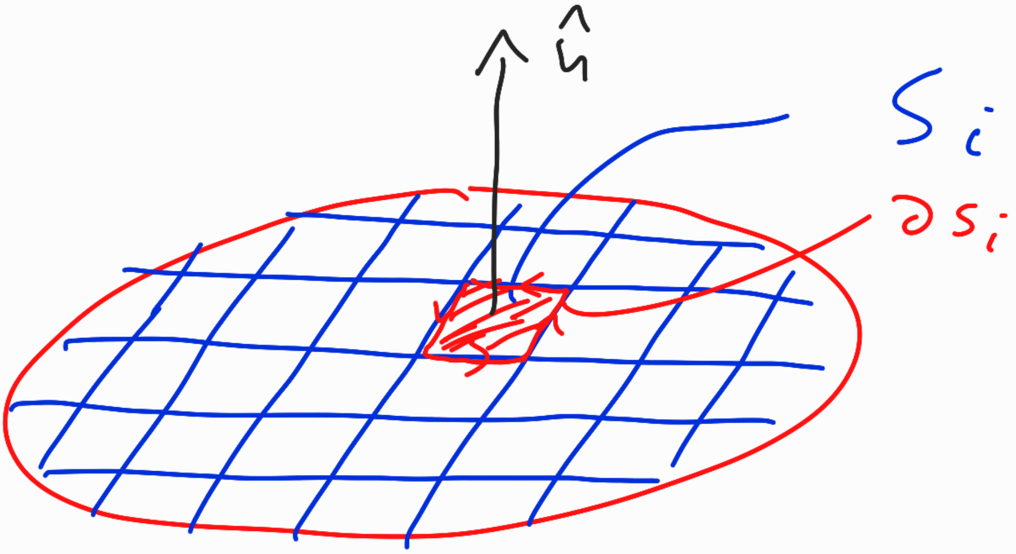


$$\int_{\partial S} \vec{A}\, d\vec{f} \overset{!}{=} \int_{\partial S_0} \vec{A}\, d\vec{f} + \int_{\partial S_1} \vec{A}\, d\vec{f}$$



$$\int_{\partial S} \vec{A}\, d\vec{f} = \int_{\partial S_{00}} \vec{A}\, d\vec{f} + \int_{\partial S_{01}} \vec{A}\, d\vec{f}$$

$$+ \ \ldots$$



$$\int_{\partial S} \vec{A}\, d\vec{f} \overset{!}{=} \sum_{i=1}^{N} \underbrace{\frac{1}{|S_i|} \int_{\partial S_i} \vec{A}\, d\vec{f} \cdot |S_i|}_{\langle \hat{u},\, \mathrm{rot}\vec{A} \rangle\, |df|}$$

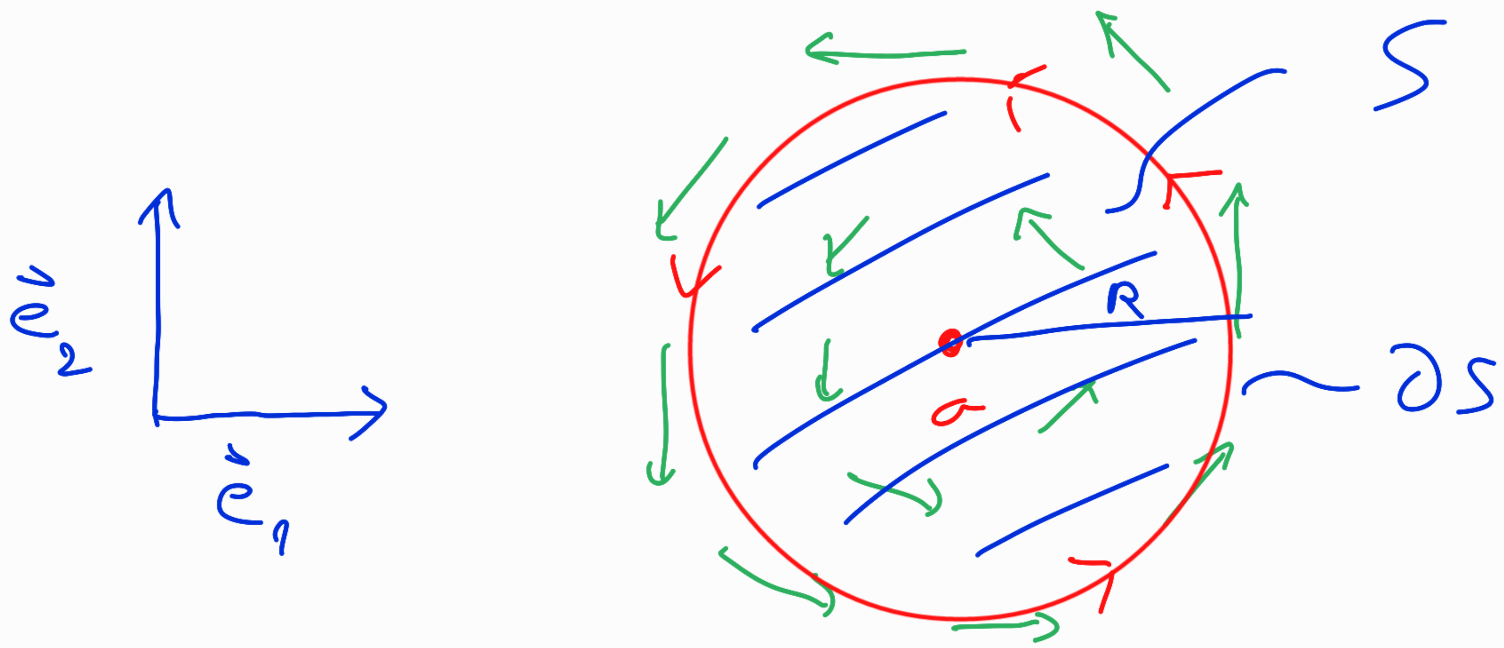
$$\underset{|S_i| \to 0}{=} \int_{S} \mathrm{rot}\vec{A}\, d\vec{f} \quad \checkmark$$

Beispiel:

$$\mathrm{rot}\,\vec{A} = 2\vec{e}_3$$

$$A(\vec{r}) = \vec{e}_3 \times \vec{r}$$

$$K_R = S$$



$$\int_{\partial K_R} \vec{A}\, d\vec{l} = 2\pi R R \overset{!}{=} \underline{\underline{2\pi R^2}}$$

L

s.v. Stokes

$$\int_{K_R} \underbrace{\mathrm{rot}\,\vec{A}}_{= 2\vec{e}_3} d\vec{f} = \underline{\underline{\pi R^2 \cdot 2}}$$