gestern: <u>Skalarprodukt</u>

$$V \times V \to \mathbb{R}$$
$$\vec{a}, \vec{b} \mapsto \langle \vec{a}, \vec{b} \rangle$$

erfüllt per def.

(SP1) $\langle \vec{a}, \vec{b} \rangle = \langle \vec{b}, \vec{a} \rangle$

(SP2) $\langle \vec{a}, \vec{a} \rangle > 0 \qquad (\vec{a} \neq \vec{0}$ sonst $= 0)$

(SP3) $\langle \vec{a}, \vec{b} + \vec{c} \rangle = \langle \vec{a}, \vec{b} \rangle + \langle \vec{a}, \vec{c} \rangle$
$\langle \vec{a}, \lambda \vec{b} \rangle = \lambda \langle \vec{a}, \vec{b} \rangle$

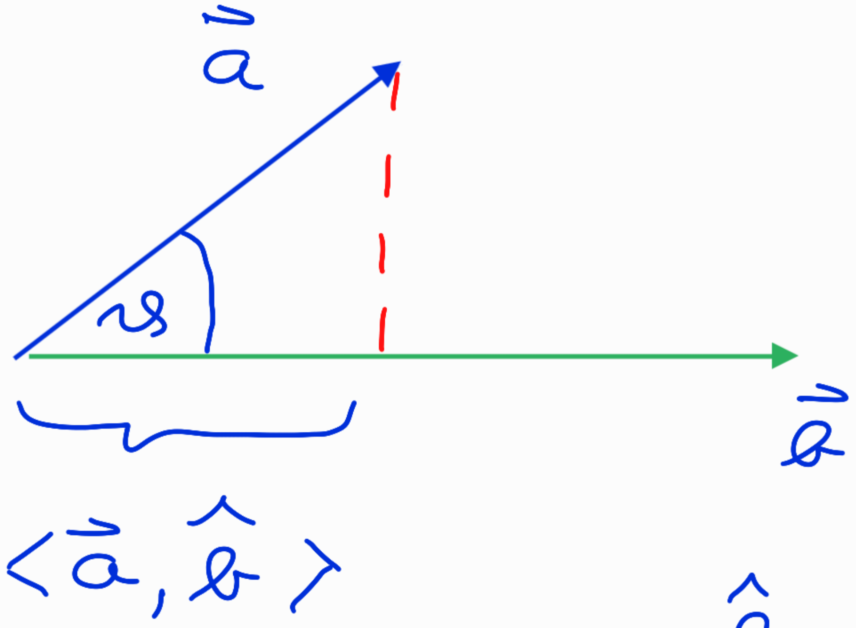
$\Rightarrow$ <u>euklidischer</u> VR !

- $|\vec{a}| := \sqrt{\langle \vec{a}, \vec{a} \rangle}$

- $\vec{a} \perp \vec{b} \quad :\Leftrightarrow \quad \langle \vec{a}, \vec{b} \rangle = 0$

- $\cos \vartheta := \langle \hat{a}, \hat{b} \rangle = \dfrac{\langle \vec{a}, \vec{b} \rangle}{|\vec{a}|\,|\vec{b}|}$

geometrische Bedeutung:



$\hat{b} = \vec{b} / |\vec{b}|$

• $\langle \vec{a}, \vec{b} \rangle = |\vec{a}|\,|\vec{b}| \cos\vartheta$

<u>Orthonormalbasis</u> (ONB):

$$B = (\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n)$$

paarweise orthogonal und vom Betrag 1

d.h. $\langle \vec{e}_i, \vec{e}_j \rangle = \delta_{ij} = \begin{cases} 1 : i = j \\ 0 : i \neq j \end{cases}$

$$\rightarrow \langle \vec{a}, \vec{b} \rangle = \sum_{i=1}^{n} a_i b_i$$

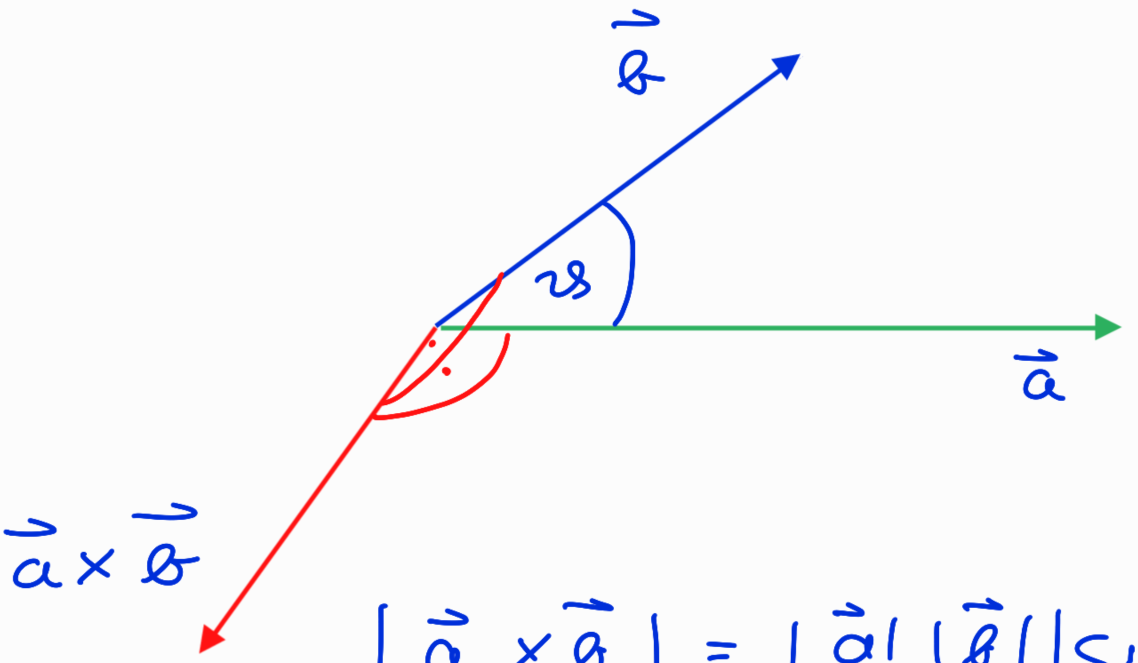
$\uparrow$

komponenten bzgl.

<u>ONB</u>

<u>heute:</u> <u>Vektorprodukt</u> für 3 dim eukl. VR

geometrisch:



$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,|\sin\vartheta|$$

Def. Vektorprodukt (Kreuzprodukt)

$$V \times V \longrightarrow V$$
$$\vec{a}, \vec{b} \longmapsto \vec{a} \times \vec{b}$$

mit definierende Eigenschaften:

(V1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (Antisymm.)

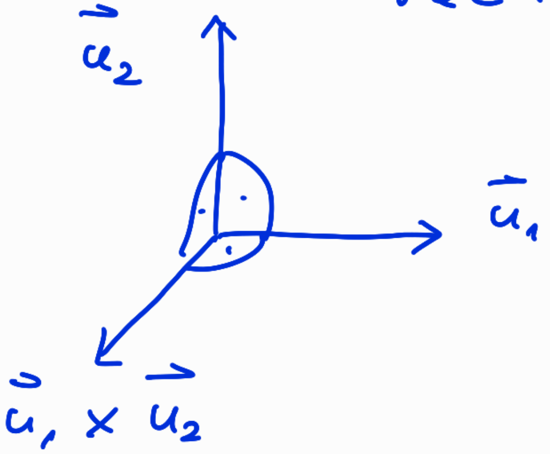
(V2) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$\vec{a} \times (\lambda \vec{b}) = \lambda (\vec{a} \times \vec{b})$

(Linearität)

(V3) $\vec{u}_1, \vec{u}_2$ orthonormal

$\Longrightarrow$ $\vec{u}_1, \vec{u}_2, \vec{u}_1 \times \vec{u}_2$ ist rechtshändige ONB

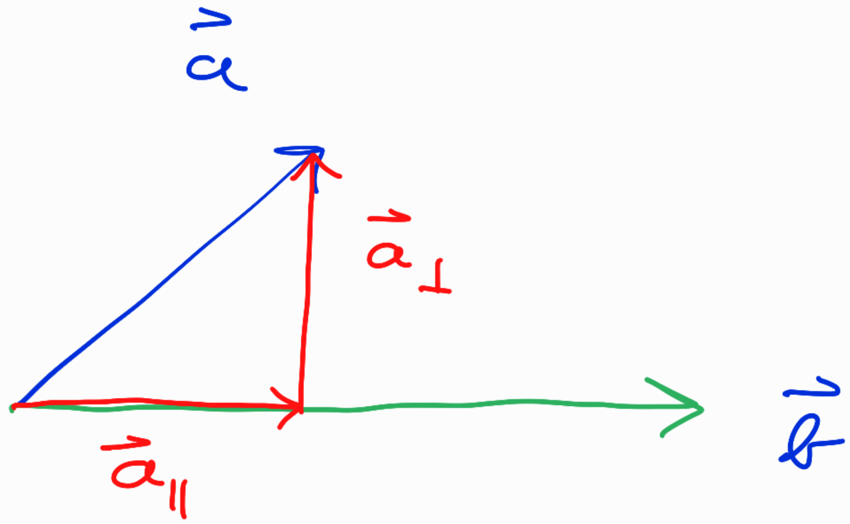


(Orthonormalität)

Eigenschaften und Anwendungen

1) $$\boxed{\vec{a} \parallel \vec{b} \iff \vec{a} \times \vec{b} = \vec{0}}$$

2) Orthogonalkomponente



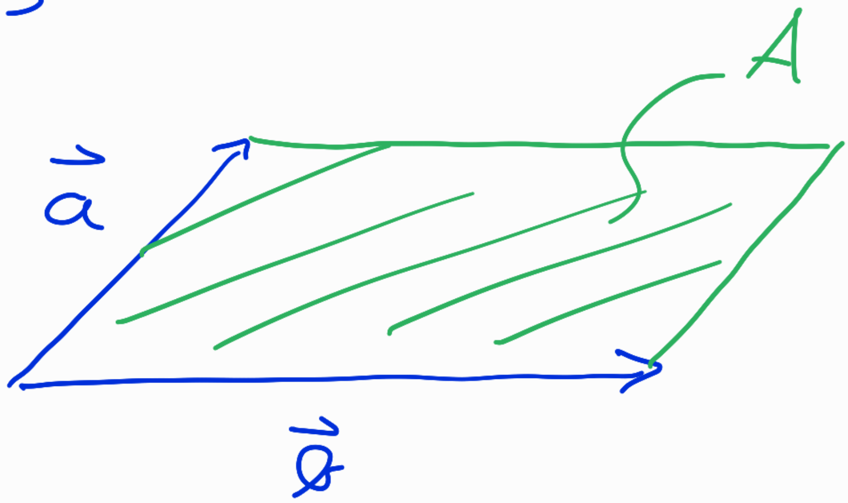
$$\vec{a}_\parallel = \langle \hat{b}, \vec{a} \rangle \hat{b}$$

$$\boxed{\vec{a}_\perp = (\hat{b} \times \vec{a}) \times \hat{b}}$$

2b)

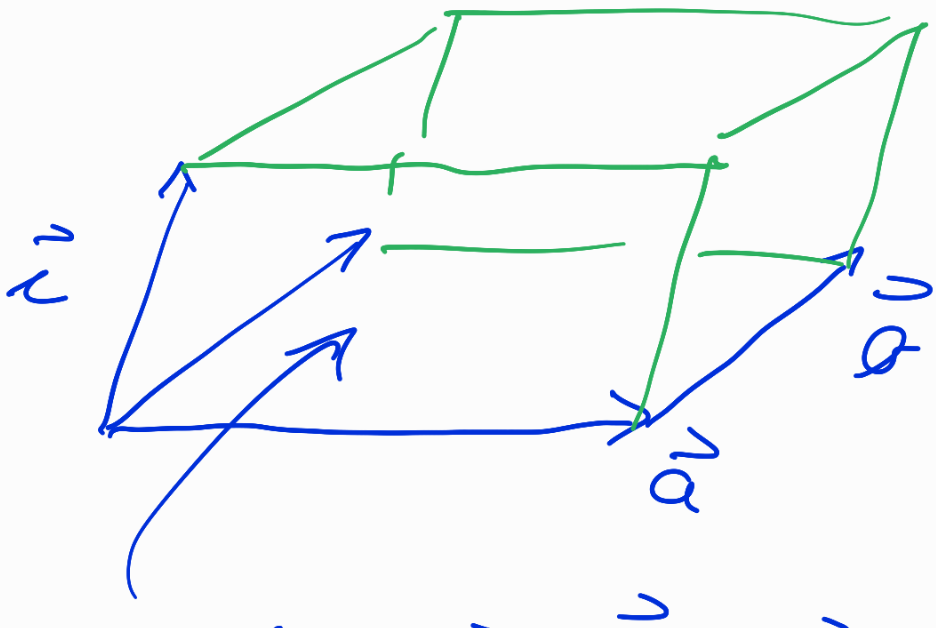
$$\boxed{|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,|\sin \vartheta|}$$

3) Flächeninhalt eines Parallelogramms



$$A = |\vec{a} \times \vec{b}|$$

4) Volumeninhalt eines Spats



$$V = |\langle \vec{a} \times \vec{b}, \vec{c} \rangle|$$

5) Berechnung im komp. bzg ONB B

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_B , \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_B$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_B \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_B = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}_B$$

---

zu 1) $\boxed{\vec{a} \parallel \vec{b} \iff \vec{a} \times \vec{b} = \vec{0}}$

„$\Rightarrow$" $\vec{a} \parallel \vec{b}$, d.h. $\vec{a} = \lambda \vec{b}$

$$\vec{a} \times \vec{b} = \vec{a} \times (\lambda \vec{a}) = \lambda (\vec{a} \times \vec{a})$$

$$= -\lambda (\vec{a} \times \vec{a}) = -\vec{a} \times \lambda \vec{a}$$

$$= -\vec{a} \times \vec{b} \curvearrowright \vec{a} \times \vec{b} = \vec{0} \,!$$

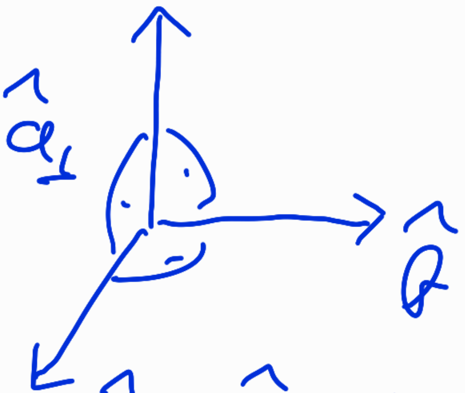
$$\text{"}\Leftarrow\text{"} \qquad \vec{a} \times \vec{b} = \vec{0} \qquad (\vec{b} \neq \vec{0})$$

$$\vec{0} = \vec{a} \times \vec{b} = (\vec{a}_\parallel + \vec{a}_\perp) \times \vec{b}$$

$$= \underbrace{\vec{a}_\parallel \times \vec{b}}_{=\vec{0}} + \vec{a}_\perp \times \vec{b}$$

$$= (|\vec{a}_\perp| \, \hat{a}_\perp) \times (|\vec{b}| \, \hat{b})$$

$$= \underbrace{|\vec{a}_\perp|}_{\overset{!}{=} 0} \, \underbrace{|\vec{b}|}_{\neq 0} \; \hat{a}_\perp \times \hat{b} \overset{!}{=} \vec{0}$$

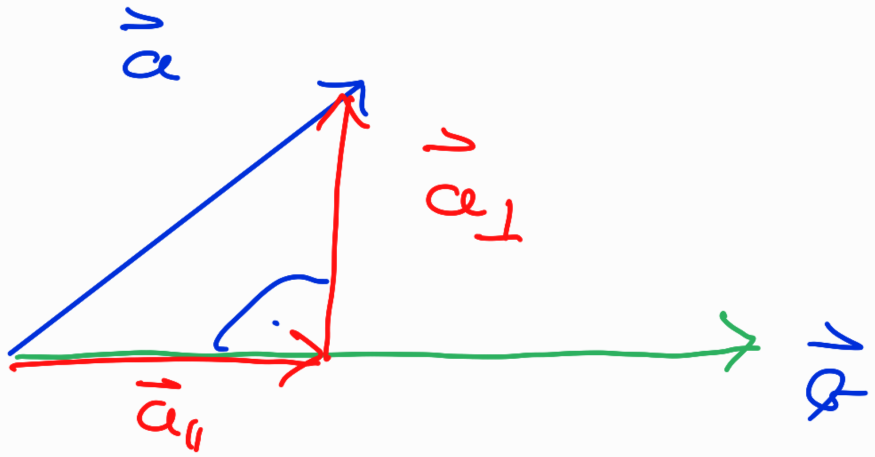


$$\hat{a}_\perp \times \hat{b} \neq \vec{0}$$

$$\longrightarrow \quad \vec{a}_\perp = \vec{0}$$

$$\vec{a} = \vec{a}_\parallel \parallel \vec{b}$$

2) Orthogonalkomponente :

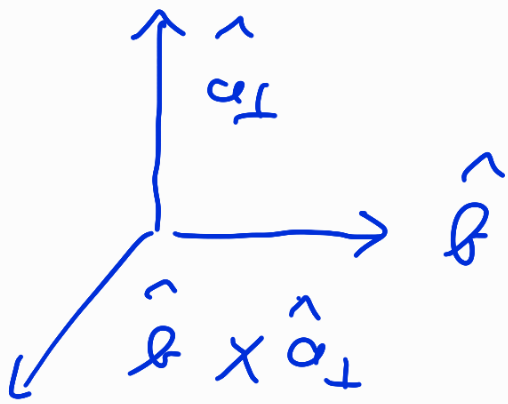


<u>Beh.</u>: $\vec a_\perp = (\hat b \times \vec a) \times \hat b$

$$\hat b \times \vec a = \hat b \times (\vec a_\parallel + \vec a_\perp)$$

$$= \underbrace{\hat b \times \vec a_\parallel}_{= \vec 0} + \hat b \times \vec a_\perp$$

$$= |\vec a_\perp| \; \hat b \times \hat a_\perp$$



$$\rightarrow |\hat{b} \times \vec{a}| = |\vec{a}_\perp|$$

$$\rightarrow (\hat{b} \times \vec{a}) \times \hat{b} = \vec{a}_\perp \quad !$$

zu 2b)

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\vartheta| \quad ✓$$

$$|\hat{b} \times \vec{a}|^2 = |\vec{a}_\perp|^2 = \underline{\underline{|\vec{a}|^2}} - \underline{\underline{|\vec{a}_\parallel|^2}}$$

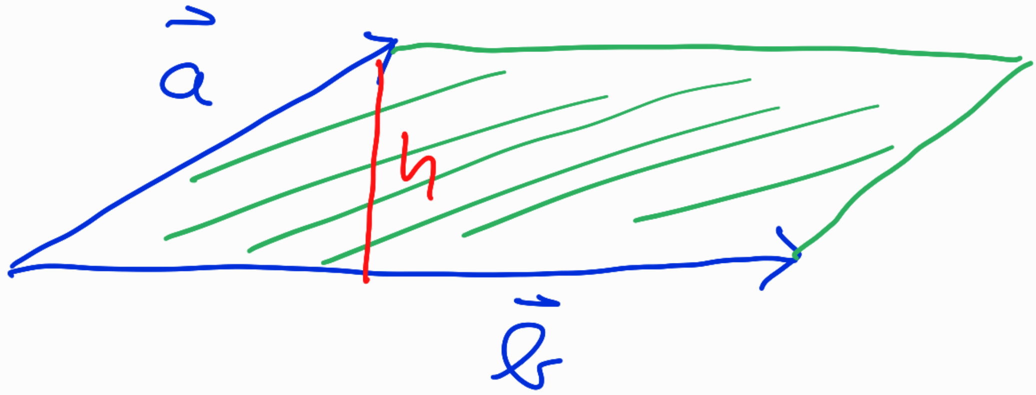
$$\left(|\vec{a}_\parallel| = |\langle \hat{b}, \vec{a} \rangle| = |\vec{a}| \cos\vartheta\right)$$

$$= |\vec{a}|^2 (1 - \cos^2\vartheta)$$

$$= |\vec{a}|^2 \sin^2\vartheta \qquad \Big| \cdot |\vec{b}|^2$$

$$\rightarrow |\vec{b} \times \vec{a}| = |\vec{a}||\vec{b}||\sin\vartheta|$$

Zu 3)



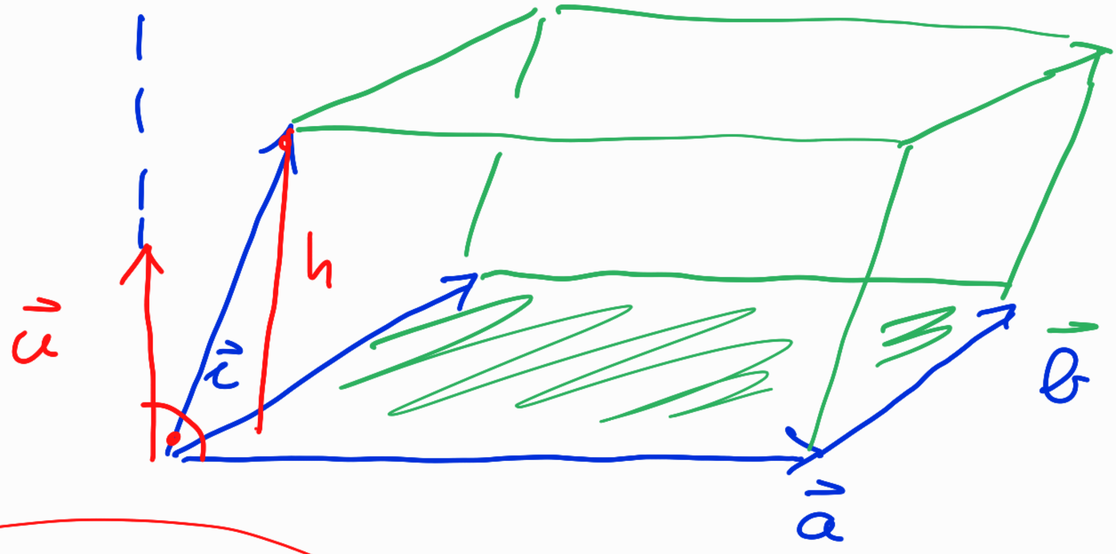
$$A = |\vec{b}| \cdot h = |\vec{b}|\,|\hat{b} \times \vec{a}|$$

$$\left\{ \begin{array}{l} h = |\vec{a}_\perp| \\ |\vec{a}_\perp| = |\hat{b} \times \vec{a}| \end{array} \right.$$

$$A = |\vec{b} \times \vec{a}| \quad !$$

Zu 4)

$|\vec{u}| = 1$;



$$\vec{u} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} \quad (*), \qquad F = |\vec{a} \times \vec{b}|$$

$$\mathbb{R} \ni h = |\langle \vec{c}, \vec{u} \rangle|$$

$$\longrightarrow \quad V = F\,h$$

$$(*)$$

$$= \cancel{|\vec{a} \times \vec{b}|}\;\left|\left\langle \vec{c}, \frac{\vec{a} \times \vec{b}}{\cancel{|\vec{a} \times \vec{b}|}} \right\rangle\right|$$

$$\boxed{V = |\langle \vec{a} \times \vec{b}, \vec{c} \rangle|}$$

<u>Spatprodukt</u> von $\vec{a}, \vec{b}, \vec{c}$

5) Berech. in kcup. bzg. ONB:

$$\vec{a} = \sum_{l=1}^{3} a_l \vec{e}_l$$

$$\vec{b} = \sum_{m=1}^{3} b_m \vec{e}_m$$

$$\underline{\vec{a} \times \vec{b}} = \Big(\sum_l a_l \vec{e}_l\Big) \times \Big(\sum_m b_m \vec{e}_m\Big)$$

$$= \sum_{l,m=1}^{3} a_l b_m \vec{e}_l \times \vec{e}_m$$

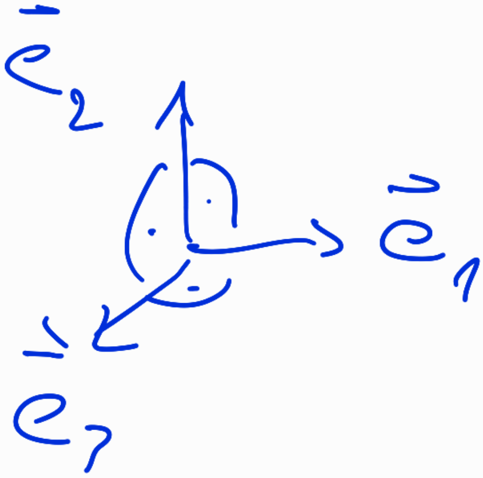
$$= \sum_{\substack{l,m=1 \\ l<m}}^{3} a_l b_m \underline{\vec{e}_l \times \vec{e}_m} + a_m b_l \underbrace{\vec{e}_m \times \vec{e}_l}_{= -\vec{e}_l \times \vec{e}_m}$$

$$= \sum_{l<m} (a_l b_m - a_m b_l)\, \vec{e}_l \times \vec{e}_m$$

$$= \underline{(a_1 b_2 - a_2 b_1)}\, \overbrace{\vec{e}_1 \times \vec{e}_2}^{= \vec{e}_3}$$

$$+ (a_1 b_3 - a_3 b_1)\, \underbrace{\vec{e}_1 \times \vec{e}_3}_{= -\vec{e}_2}$$

$$+ \underline{(a_2 b_3 - a_3 b_2)}\, \underbrace{\vec{e}_2 \times \vec{e}_3}_{= \vec{e}_1}$$



$$= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}_B$$