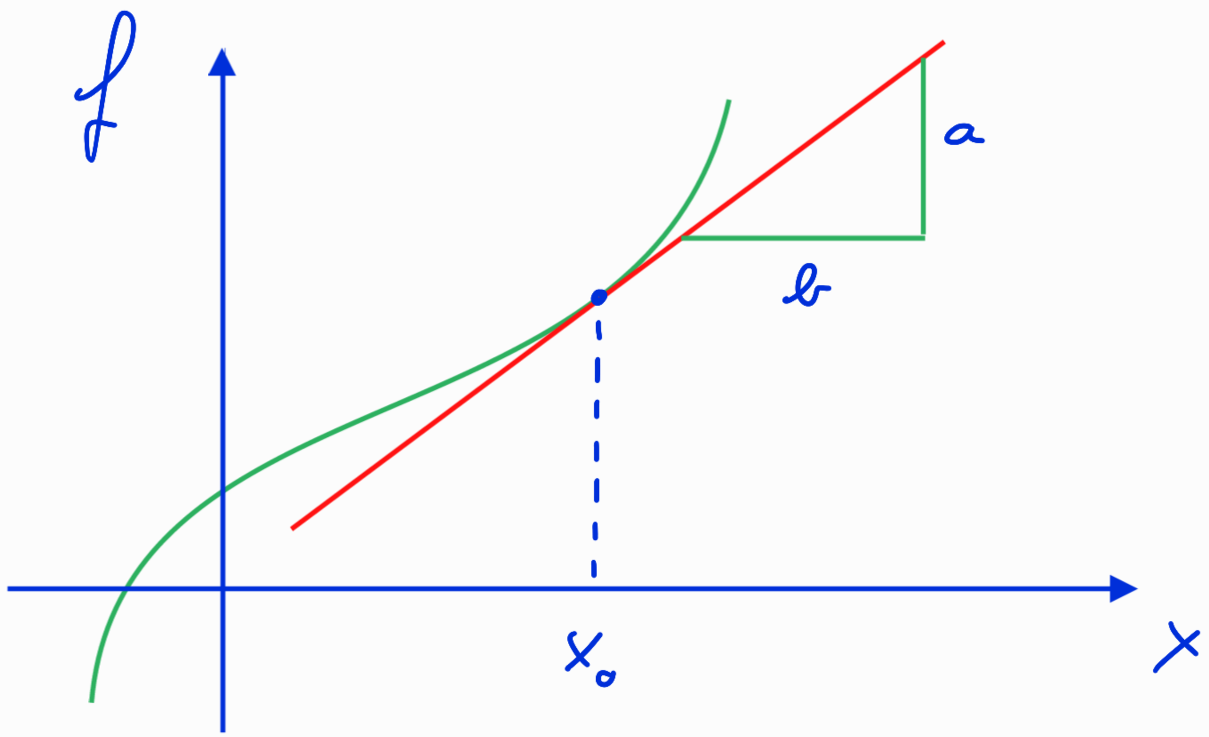
Letzte Vrlsg.:

- Funktion: $f: D \to W$
  $x \mapsto f(x) = [\dots]$

- $f+g, \ \lambda f, \ f \cdot g, \ f/g, \ f \circ g, \ f^{-1}$

Ableitung $f'(x_0)$ von f in x_0

= Steigung der Tangente des Funktionsgraphen von f in x_0



$$\boxed{f'(x_0) = \frac{a}{b} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

Ableitung von $f: D \to \mathbb{R}$

$$= \text{Funktion } f': \begin{array}{l} D \to \mathbb{R} \\ x \mapsto f'(x) \end{array}$$

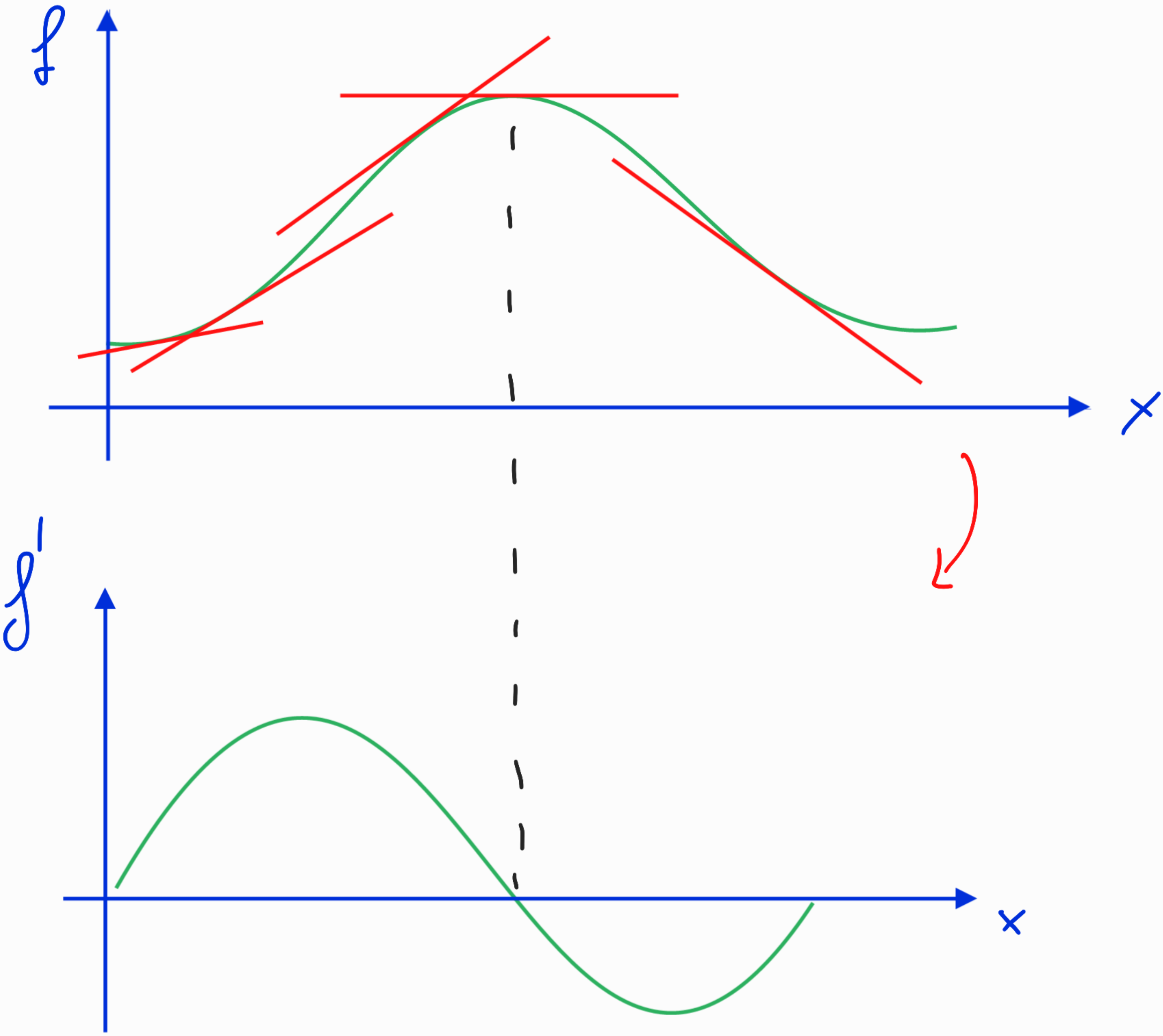
alt. Notation:

$$f' = \frac{df}{dx} \quad ; \quad f'(x) = \frac{df}{dx}(x)$$

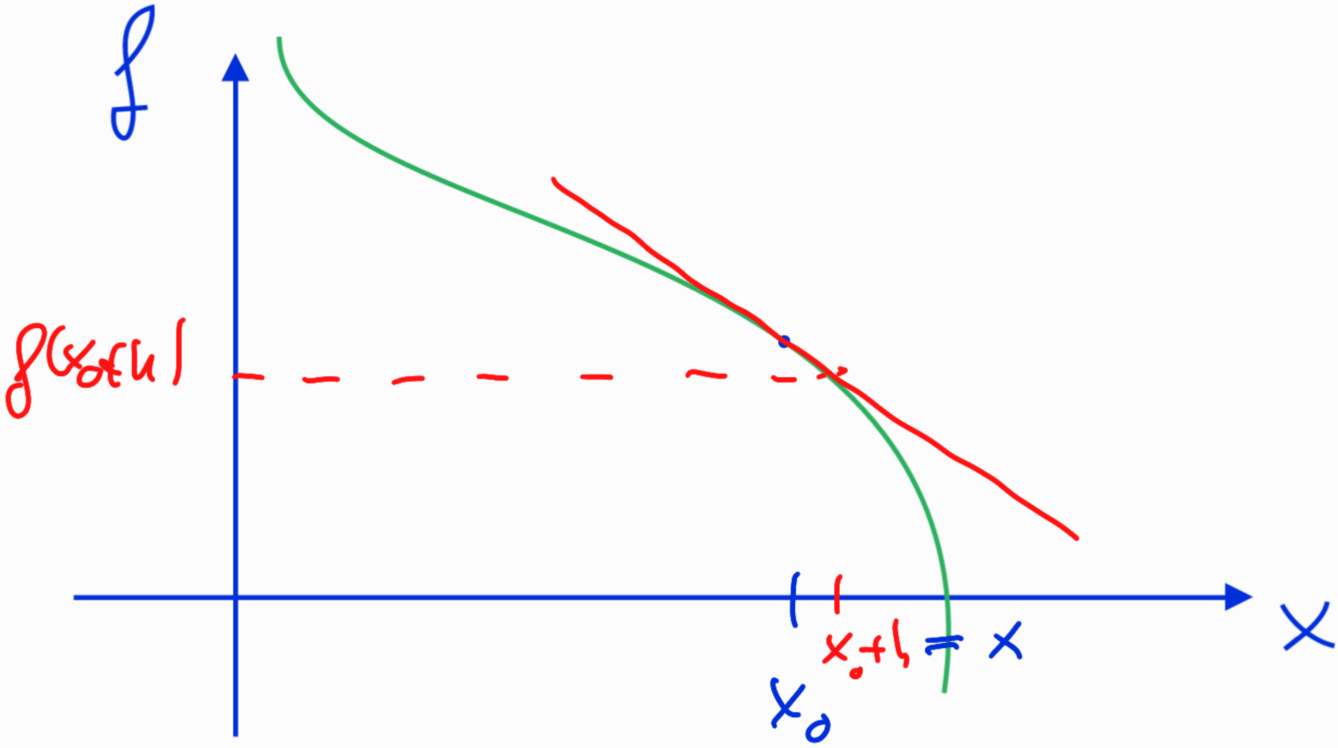
Ableitung nach Zeit t:

$$f'(t) = \dot{f}(t)$$

Beispiel:



Lineare Näherung:



$$f'(x_0) \approx \frac{1}{h}\big(f(x_0+h) - f(x_0) \big) \quad !$$

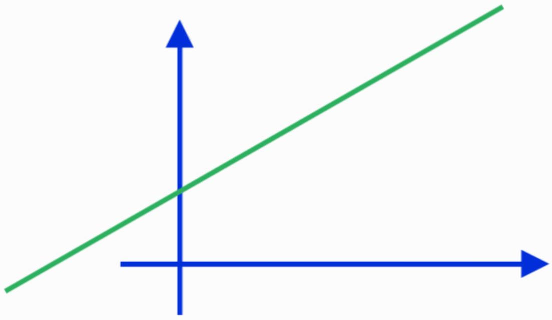
$$\boxed{f(x_0+h) = f(x_0) + f'(x_0)\,h} \qquad !$$

≙ lineare Näherung von f in x_0
(gültig für hinreichend kleine h)

$$\big(p(x) = f(x_0) + f'(x_0)(x-x_0) \big)$$

Elementare Bestimmung der Ableitung:

1) $f(x) = ax + b$



$$f'(x) = a \quad \left(= \frac{1}{h}\left(\cancel{a(x}+h) \cancel{+b} - \cancel{ax} \cancel{-b} \right) \right) \checkmark$$

2) $g(x) = x^2 \quad , \quad g'(x) = 2x$

$$g'(x) = \frac{1}{h}\left(g(x+h) - g(x) \right)$$

$$= \frac{1}{h}\left((x+h)^2 - x^2 \right)$$

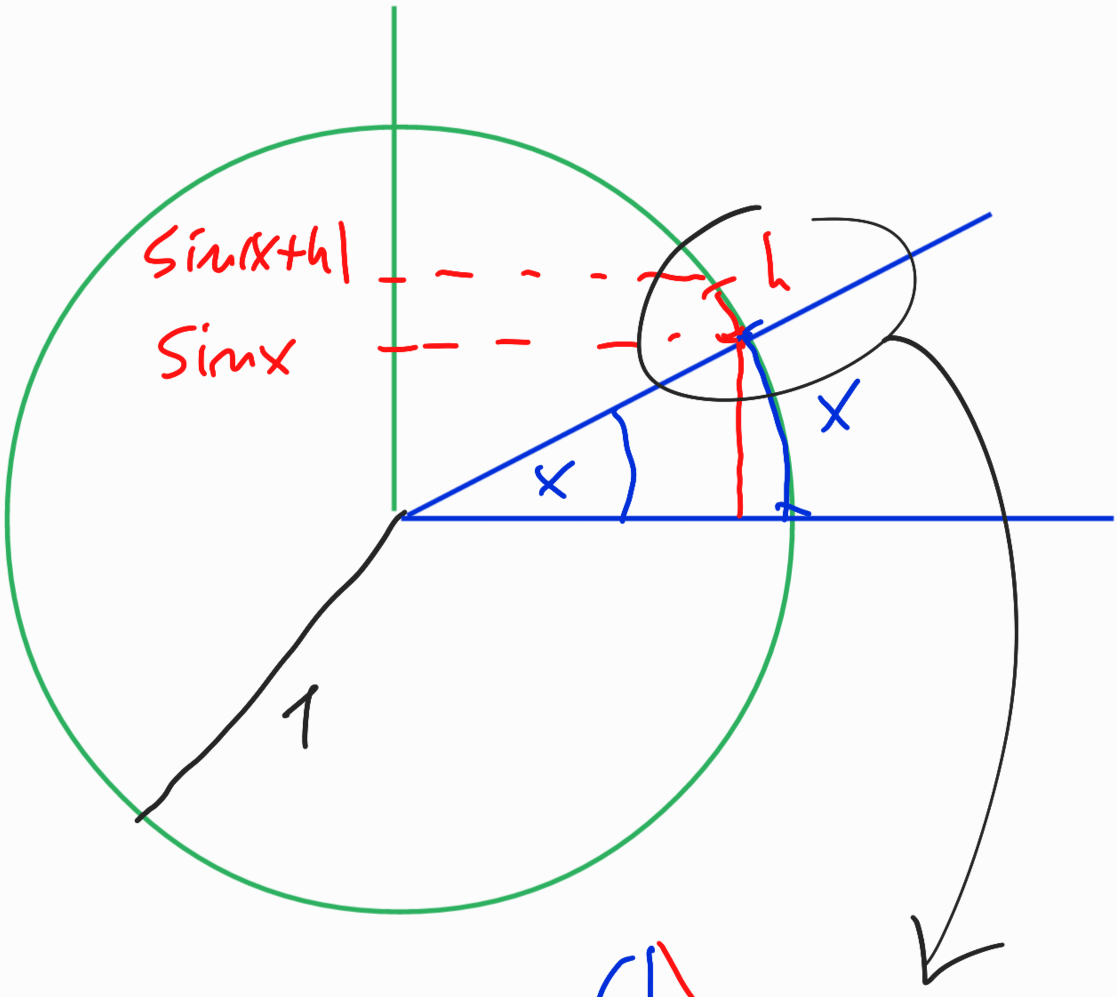
$$= \frac{1}{\cancel{h}}\left(\cancel{x^2} + 2x\cancel{h} + h^2 - \cancel{x^2} \right)$$

$$= 2x + h \xrightarrow[h \to 0]{} 2x \quad !$$

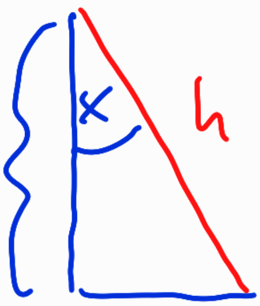
$$\boxed{(x^2)' = 2x}$$

analog: $\boxed{(x^n)' = n x^{n-1}}$

3) $(\sin x)' = ?$



$\sin(x+h) - \sin x$



$\approx h \cos x$

$$(\sin x)' = \frac{1}{h}\left(\sin(x+h) - \sin x\right) \approx \cos x$$

$$\boxed{(\sin x)' = \cos x}$$

analog: $\boxed{(\cos x)' = -\sin x}$

4) $f(x) = a^x \quad ; \quad a \in \mathbb{R}_+$

$$f'(x) = \frac{1}{h}\left(a^{x+h} - a^x \right)$$

$$= \underbrace{\frac{1}{h}\left(a^h - 1 \right)}_{\text{unabhängig von } x\,!} \cdot a^x$$

$\rightsquigarrow \quad f'(x) = \ell_a \; a^x$

d.h. $\boxed{(a^x)' = \ell_a \, a^x}$

mit $\ell_a = \lim\limits_{h \to 0} \frac{1}{h}\left(a^h - 1 \right)$

<u>Euler</u>: wähle a so, dass

$\ell_a = 1$!

$\rightarrow$ Eulersche Zahl e :

bestimmt durch

$$\lim_{h\to 0} \frac{1}{h}\left(e^h - 1\right) \stackrel{!}{=} 1$$

statt $h \to 0$: $h = \frac{1}{n}$, $n \in \mathbb{N}$, $n \to \infty$

$$n\left(e^{1/n} - 1\right) \stackrel{!}{=} 1 \quad (n \to \infty)$$

$$\boxed{e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n}$$

n	1	2	3	. . . . .	1000
$\left(\frac{n+1}{n}\right)^n$	2	2,25	2,37		2,718...

morgen:

$$e = \lim_{n \to \infty} \left(\frac{n+1}{n} \right)^n = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$+ \ldots$$

$$\overset{!}{=} 2{,}718281828\ldots$$

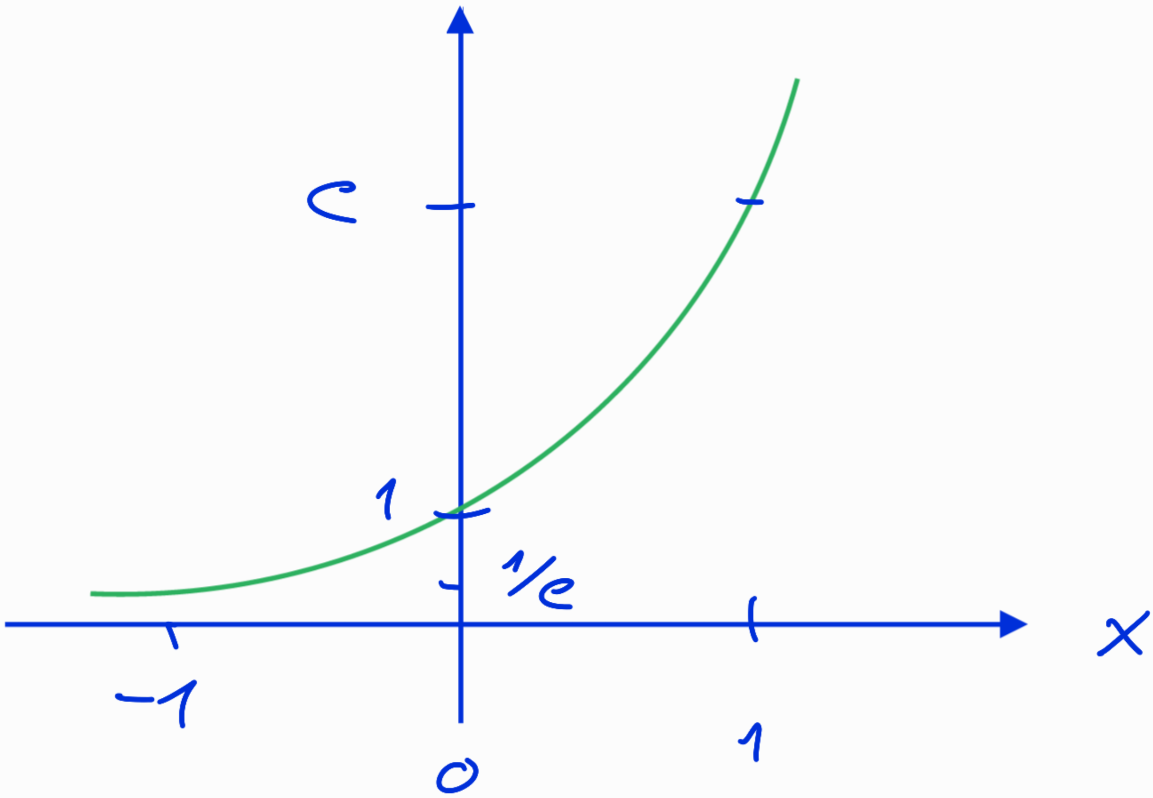
$$\rightsquigarrow \boxed{(e^x)' = e^x}$$

Exkurs: Exponentialfunktion

$$\exp : \mathbb{R} \longrightarrow \mathbb{R}_+$$

$$x \longmapsto \exp(x) = e^x$$

$$\boxed{\exp' = \exp} \quad !$$



- $e^{a+b} = e^a e^b$
- $(e^a)^\lambda = e^{\lambda a}$

Ableitungsregeln

1)
$$(f+g)' = f' + g' \qquad \text{(Linearität)}$$
$$(\lambda f)' = \lambda f'$$

2)
$$(fg)' = f'g + fg' \qquad \text{(Produktregel)}$$

3)
$$(f/g)' = (f'g - fg')/g^2 \qquad \text{(Quotientenregel)}$$

4)
$$(f \circ g)' = (f' \circ g) \cdot g' \qquad \text{(Kettenregel)}$$

$$f(g(x))' = \underbrace{f'(g(x))}_{\text{äußere}} \cdot \underbrace{g'(x)}_{\text{innere}} \quad \text{Ableitung}$$

5)
$$\left(f^{-1}(y)\right)' = \frac{1}{f'(f^{-1}(y))}.$$

Kettenregel ?

$$(f\circ g)(x+a) = f(\underbrace{g(x+a)}_{g(x)+g'(x)\cdot a \ \text{(l.N.)}})$$

$$= f(g(x) + g'(x)a)$$

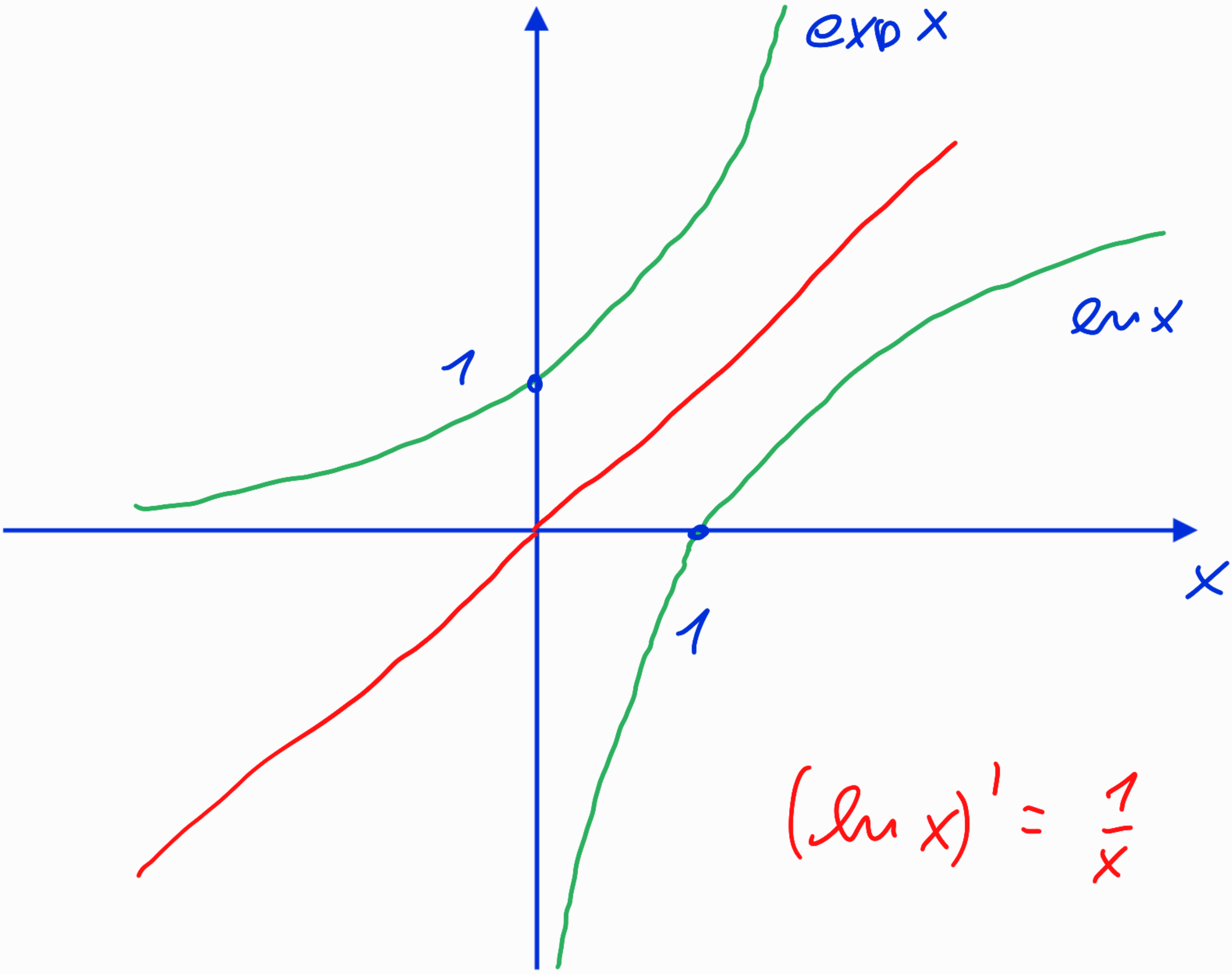
$$\overset{\text{l.N.}}{=} \underbrace{f(g(x))}_{\| } + \underbrace{f'(g(x))\,g'(x)\,a}_{\|}$$

$$= (f\circ g)(x) + (f\circ g)'(x)\,a$$

Exkurs: natürliche Logarithmus ln

= Umkehrfkt der Fkt. exp !

$$\boxed{\begin{aligned} \ln := \exp^{-1} : \mathbb{R}_+ &\to \mathbb{R} \\ x &\mapsto \ln(x) = \exp^{-1}(x) \end{aligned}}$$



$$(\ln x)' = \frac{1}{x}$$

- $\ln(ab) = \ln(a) + \ln(b)$
- $\ln(a^\lambda) = \lambda \ln(a)$

$$\ln(ab) = \ln(e^{\ln a} e^{\ln b})$$

$$= \ln(e^{\ln a + \ln b})$$

$$= \ln a + \ln b \,;$$

$$\ln(a^\lambda) = \ln((e^{\ln a})^\lambda)$$

$$= \ln(e^{\lambda \ln a}) = \lambda \ln a$$

Ableitung:

$$(\ln x)' = ?$$

$$(\exp \circ \ln)(x) \overset{!}{=} x$$

$$e^{\ln x} = x \quad \Big| \frac{d}{dx}$$

$$\underbrace{e^{\ln x}}_{= x\,!} \cdot (\ln x)' = 1$$

äußere Abl.

$(\exp' = \exp)$

$$\Rightarrow \quad \boxed{(\ln x)' = \frac{1}{x}}$$

4) allg. Potenzfkt. :

$$(\ldots)^\alpha : \boxed{\begin{array}{l} \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \\ x \longmapsto x^\alpha := e^{\alpha \ln x} \end{array}}$$

$\boxed{\alpha \in \mathbb{R}}$

$$\boxed{(x^\alpha)' = \alpha x^{\alpha-1}}$$

(genau wie $(x^n)' = n x^{n-1}$ $n \in \mathbb{Z}$)

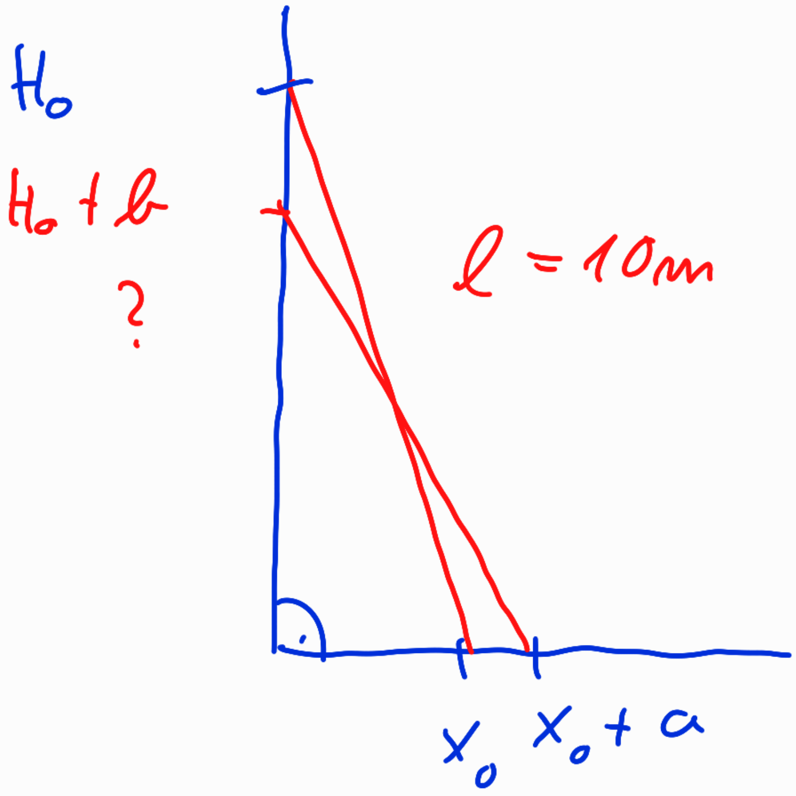
$\ulcorner$ $(x^\alpha)' = (e^{\alpha \ln x})'$

$= e^{\alpha \ln x} \underbrace{\alpha (\ln x)'}$

$= x^\alpha \quad \alpha / x$

$= \alpha x^{\alpha - 1}$ $\lrcorner$

Anwendungsbeispiel



$$H^2 + x^2 = \ell^2$$

$$\rightarrow \quad H = (\ell^2 - x^2)^{1/2}$$

$$H(x) = (\ell^2 - x^2)$$

$$H(x_0 + a) = H(x_0) + \underbrace{H'(x_0)\, a}_{= e\,!}$$

lin. N.

$$H'(x) = \left((l^2 - x^2)^{1/2} \right)'$$

$$= \frac{1}{\cancel{2}} (l^2 - x^2)^{\frac{1}{2} - 1} \cdot (-\cancel{2}x)$$

$$= -\frac{1}{\sqrt{l^2 - x^2}} \, x =$$

$$\rightarrow \quad b = -\frac{x_0}{\sqrt{l^2 - x_0^2}} \cdot a$$

$$\boxed{b = -\frac{x_0 \, a}{H_0}}$$