Letzte Vrlsg.: Taylor-Entwicklung

- $$\boxed{\tilde{f}_n(x) = \sum_{\ell=0}^{n} \frac{1}{\ell!} f^{(\ell)}(0)\, x^\ell}$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \dots$$

- $$\widetilde{\exp}_n(x) = \sum_{\ell=0}^{n} \frac{1}{\ell!} x^\ell$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4} + \dots$$

konvergent für $n \to \infty$

$\Rightarrow$ Potenzreihe der Exponentialfkt.:

- $$\boxed{\exp(x) = \sum_{\ell=0}^{\infty} \frac{x^\ell}{\ell!}}$$

- $\left(\frac{n+1}{n}\right)^n \xrightarrow[n\to\infty]{} e = \exp(1)$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \dots$$

$$= 2{,}7182\dots$$

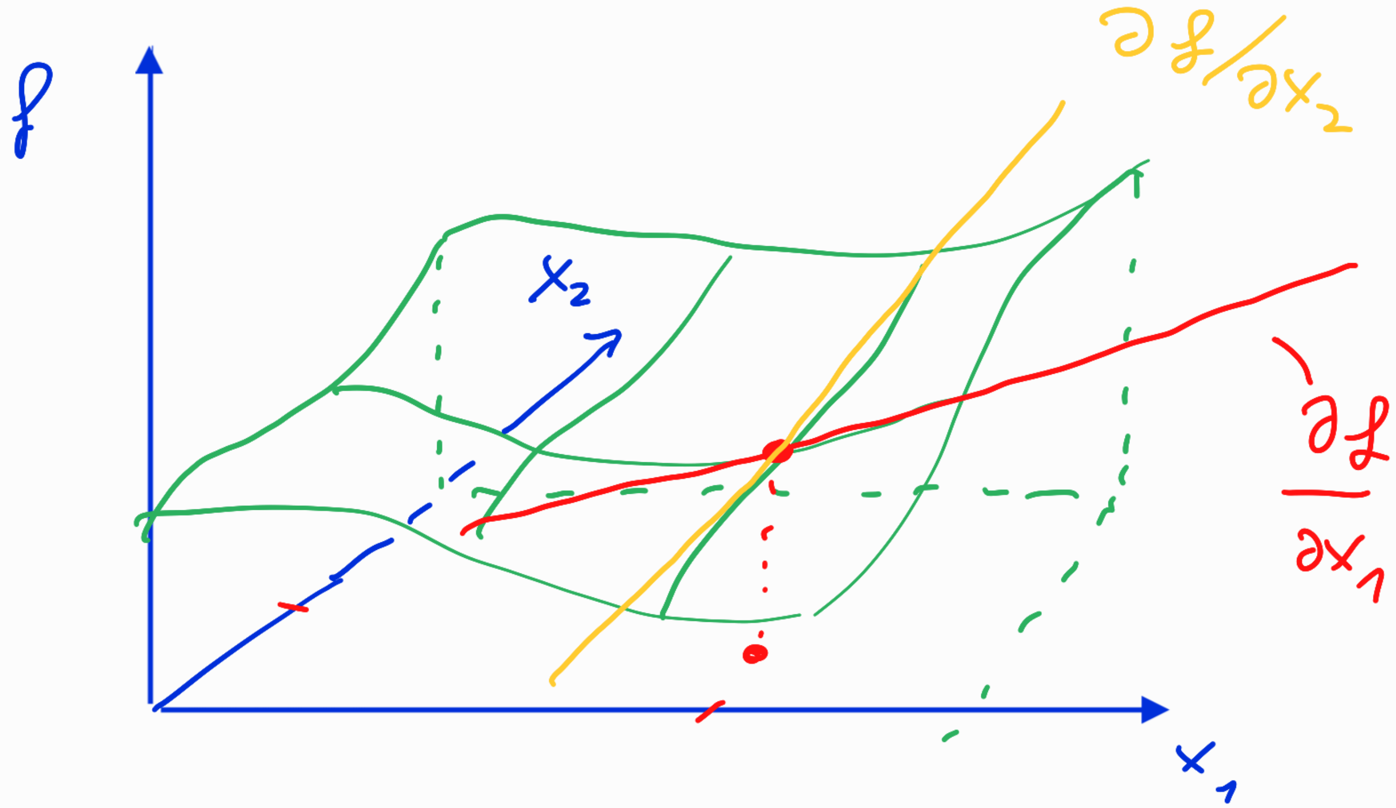
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

$$= \sum_{l=0}^{\infty} x^l \qquad (|x|<1)$$

Partielle Ableitung und Gradient einer Fkt. f in n Variablen

$$f : D \to \mathbb{R} \qquad D \subset \mathbb{R}^n$$

$$\vec{x} \mapsto f(\vec{x}) = f(x_1, x_2, \dots, x_n)$$



Partielle Ableitung von f in $\vec{x}$ nach x_ℓ

$$\frac{\partial f}{\partial x_\ell}(\vec{x}) = \Big(f(\underbrace{x_1, \ldots, x_{\ell-1}}_{\text{fest}}, \underbrace{x_\ell}_{\text{variabel}}, \underbrace{x_{\ell+1} \ldots}_{\text{fest}}) \Big)'$$

$$\boxed{\frac{\partial f}{\partial x_\ell}(\vec{x}) = \lim_{h \to 0} \frac{1}{h} \Big(f(\vec{x} + h\vec{e}_\ell) - f(\vec{x}) \Big)}$$

Beispiele: 1) $f(x_1, x_2) = x_1^2 x_2$

$$\Rightarrow \frac{\partial f}{\partial x_1} = 2 x_1 x_2, \quad \frac{\partial f}{\partial x_2} = x_1^2$$

2) $g(x, y) = \sin(x) \cdot \cos(y)$

$$\Rightarrow \frac{\partial g}{\partial x} = \cos(x) \cdot \cos(y); \quad \frac{\partial g}{\partial y} = -\sin(x) \sin(y)$$

3) Betragsfkt: $\vec{x} \longmapsto |\vec{x}|$

$$|\vec{x}| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

($\mathbb{R}^n$ mit Standard-S.P.:

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i y_i \quad)$$

$$\frac{\partial |\vec{x}|}{\partial x_e} = \frac{1}{\cancel{2}\sqrt{\sum_i x_i^2}} \cancel{2} x_e = \frac{x_e}{|\vec{x}|}$$

4) $f(\vec{x}) = h(|\vec{x}|)$; $h: \mathbb{R} \to \mathbb{R}$

$$\frac{\partial f(\vec{x})}{\partial x_e} = \frac{\partial h(|\vec{x}|)}{\partial x_e} = h'(|\vec{x}|) \frac{x_e}{|\vec{x}|}$$

Lineare Näherung

$n=1$: $\quad f(x+h) = f(x) + f'(x)\,h$

allg. n:

$$f(\vec{x}+\vec{h}) = f(x_1+h_1, x_2+h_2, \dots, x_n+h_n)$$

$$= f(\vec{x}) + \frac{\partial f}{\partial x_1}(\vec{x})\,h_1 + \frac{\partial f}{\partial x_2}(\vec{x})\,h_2 + \dots + \frac{\partial f}{\partial x_n}(\vec{x})\,h_n$$

d.h.

$$\boxed{f(\vec{x}+\vec{h}) = f(\vec{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x})\,h_i}$$

$\leadsto$ Gradient von f

<u>Def</u>: Gradient von f in $\vec{x}$:

$$\boxed{\operatorname{grad} f(\vec{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x})\, \vec{e}_i}$$

$B = (\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n)$

in Komponenten:

$$\operatorname{grad} f(\vec{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\vec{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\vec{x}) \end{pmatrix}_B$$

$\rightsquigarrow$ Lineare Näherung:

$$\boxed{f(\vec{x}+\vec{h}) = f(\vec{x}) + \langle \operatorname{grad} f(\vec{x}), \vec{h} \rangle}$$

Notation: Nabla-Operator

$$\vec{\nabla} = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \vdots \\ \partial/\partial x_n \end{pmatrix}$$

$$\Rightarrow \vec{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \text{grad } f$$

$$\boxed{\text{grad } f(\vec{x}) = \vec{\nabla} f(\vec{x})}$$

$\ulcorner$ $n = 3$; $x_1 = x$, $x_2 = y$, $x_3 = z$

$$\vec{\nabla} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$$

$\lrcorner$

<u>Beispiele</u>: 1) $f(x_1, x_2) = x_1^2 x_2$

$$\text{grad } f(\vec{x}) = \begin{pmatrix} 2x_1 x_2 \\ x_1^2 \end{pmatrix}$$

2) $g(x, y) = \sin x \cdot \cos y$

$$\rightarrow \text{grad } g(\vec{x}) = \begin{pmatrix} \cos x \cos y \\ -\sin x \sin y \end{pmatrix}$$

3) $$\frac{\partial |\vec{x}|}{\partial x_e} = \frac{x_e}{|\vec{x}|}$$

$$\rightsquigarrow \text{grad } |\vec{x}| = \frac{1}{|\vec{x}|} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{\vec{x}}{|\vec{x}|}$$

$$\boxed{\text{grad } |\vec{x}| = \hat{x}}$$

$n = 3$; $\vec{x} = \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $|\vec{r}| = r$

$$\text{grad } r = \hat{r} = \frac{\vec{r}}{r}$$

4) $$f(\vec{x}) = h(|\vec{x}|)$$

$$\frac{\partial f(\vec{x})}{\partial x_\ell} = h'(|\vec{x}|)\, x_\ell / |\vec{x}|$$

$$\hookrightarrow \boxed{\operatorname{grad} h(|\vec{x}|) = h'(|\vec{x}|)\, \hat{x}}$$

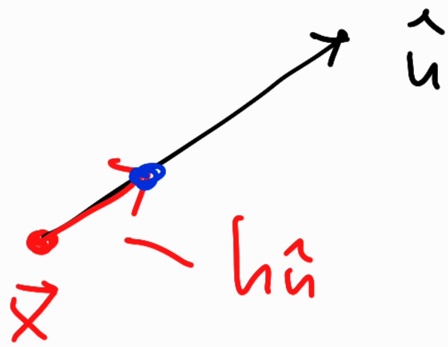
$n = 3$:

$$\boxed{\operatorname{grad} h(r) = h'(r) \cdot \hat{r}}$$

<u>Richtungsableitung</u> von f in $\vec{x}$

in Richtung $\hat{u}$ $\quad \left(|\hat{u}| = 1, \; \hat{u} \in \mathbb{R}^n \right)$

$$\partial_{\hat{u}} f(\vec{x}) = \lim_{h \to 0} \frac{1}{h} \left(f(\vec{x} + h\hat{u}) - f(\vec{x}) \right)$$



$h \ll 1$

Lin. Näherung:

$$\frac{1}{h} \left(f(\vec{x} + h\hat{u}) - f(\vec{x}) \right)$$

$$= \frac{1}{\cancel{h}} \langle \operatorname{grad} f(\vec{x}), \cancel{h}\hat{u} \rangle$$

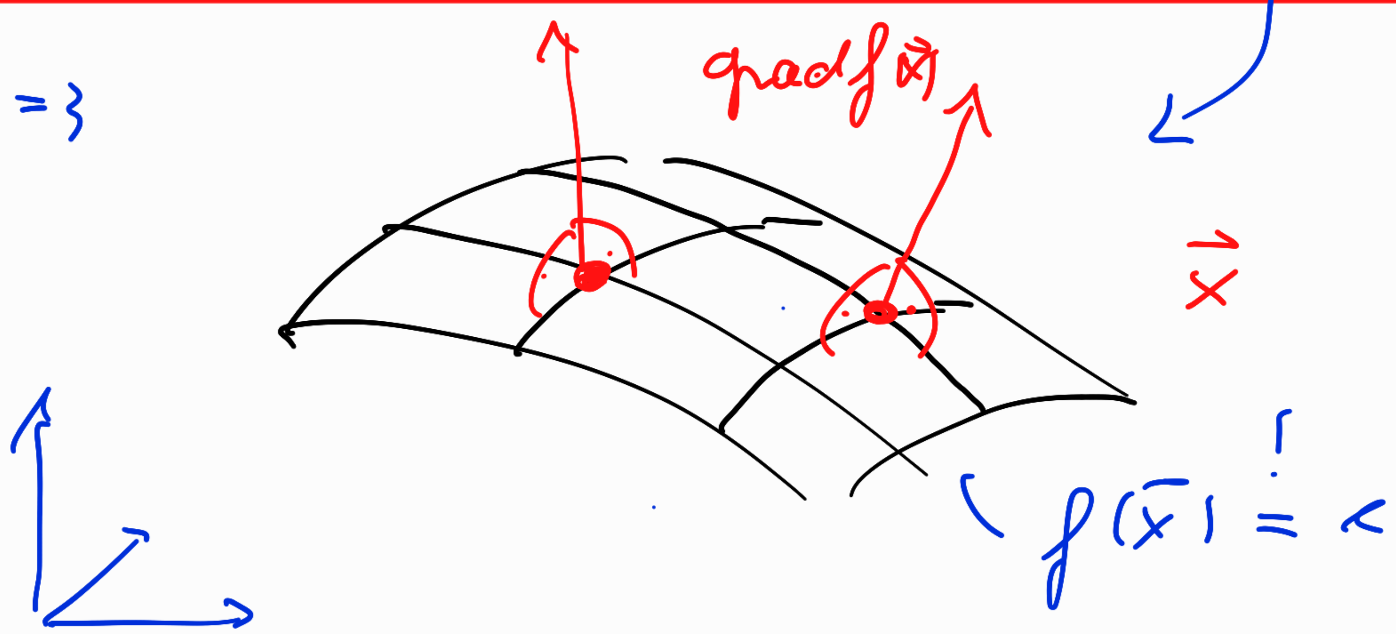
$$\boxed{\partial_{\hat{u}} f(\vec{x}) = \langle \operatorname{grad} f(\vec{x}), \hat{u} \rangle}$$

Eigenschaften des Gradienten

1) $\operatorname{grad} f(\vec{x}) \parallel$ Richtung des stärksten Anstiegs von f in $\vec{x}$

2) $\operatorname{grad} f(\vec{x}) \perp$ Hyperfläche $f(\vec{x}) = c$

$n = 3$

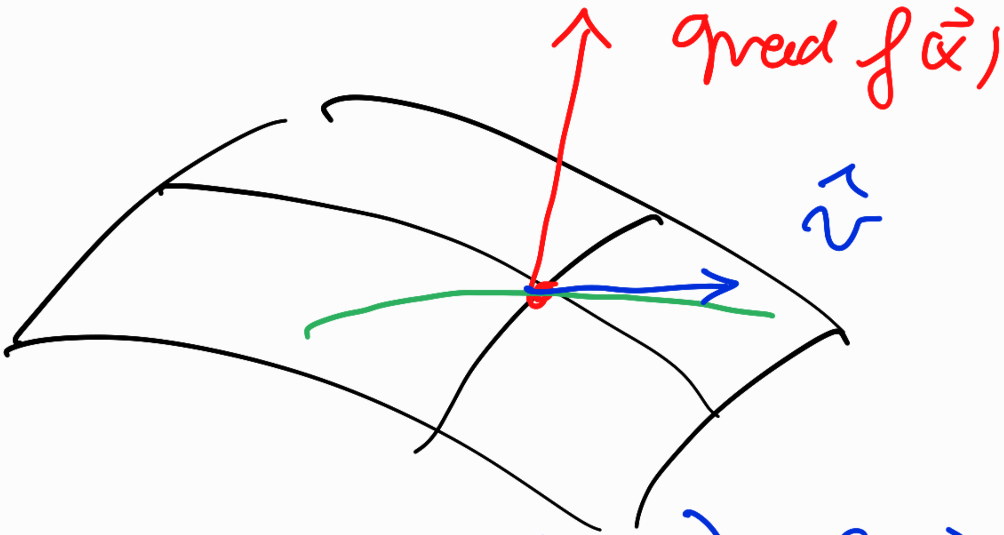


zu 1)

$$\partial_{\hat{u}} f(\vec{x}) = \langle \operatorname{grad} f(\vec{x}), \hat{u} \rangle$$

maximal für $\hat{u} \parallel \operatorname{grad} f(\vec{x})$!

zu 2)



$\hat{v}$ tangential

$$\Leftrightarrow \quad \partial_{\vec{v}} f(\vec{x}) \overset{!}{=} 0$$

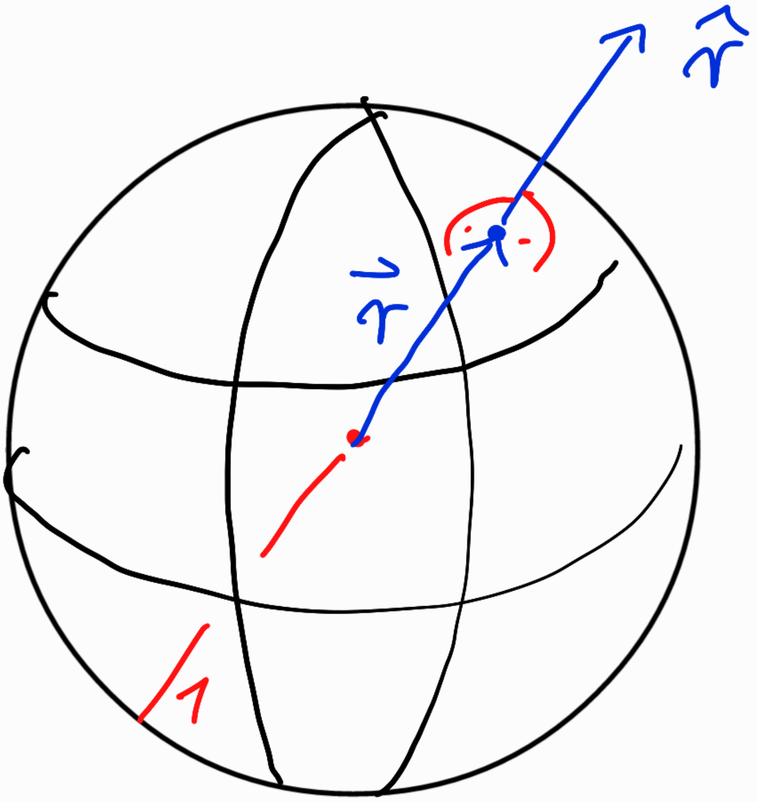
$$\Leftrightarrow \quad \langle \operatorname{grad} f(\vec{x}), \vec{v} \rangle = 0$$

$$\Leftrightarrow \quad \operatorname{grad} f(\vec{x}) \perp \vec{v}$$

Beispiel: $n = 3$, $\vec{x} = \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$f(\vec{r}) = |\vec{r}| = r$$

$$f(\vec{r}) \stackrel{!}{=} R = 1 \quad ; \quad \operatorname{grad} r = \hat{r}$$



Höhere partielle Ableitungen:

2. part. Ableitungen: $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^n$

$$\boxed{\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}) := \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}(\vec{x})\right)}$$

Beispiele: $f(x_1, x_2) = x_1^2 x_2$

$$\frac{\partial^2 f}{\partial x_1 \partial x_1} \equiv \frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right)$$

$$= \frac{\partial}{\partial x_1}(2 x_1 x_2) = 2 x_2$$

$$\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial}{\partial x_2}\underbrace{\left(\frac{\partial}{\partial x_2} x_1^2 x_2\right)}_{= x_1^2} = 0$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\underbrace{\left(\frac{\partial}{\partial x_2} x_1^2 x_2\right)}_{x_1^2} = 2x_1$$

$$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2}\underbrace{\left(\frac{\partial}{\partial x_1} x_1^2 x_2\right)}_{2x_1 x_2} = 2x_1$$

Satz von Schwarz: (f hinreichend glatt)

$$\boxed{\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}}$$

$(m+1)$-te part. Ableitung: $\mathbb{R}^n$

$$\frac{\partial^{m+1} f(\vec{x})}{\partial x_{l_1} \partial x_{l_2} \cdots \partial x_{l_{m+1}}}$$

$$:= \frac{\partial}{\partial x_{l_{m+1}}} \left(\frac{\partial^m f(\vec{x})}{\partial x_{l_1} \cdots \partial x_{l_m}} \right)$$

→ Taylor-Entwicklung in n Variablen:

$$f : D \to \mathbb{R} \qquad D \subset \mathbb{R}^n$$

in $\vec{x}_0 = \vec{0}$

$$\tilde{f}(\vec{x}) = f(\vec{0}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{0}) \cdot x_i$$

$$+ \frac{1}{2!} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{0}) x_i x_j$$

$$+ \frac{1}{3!} \sum_{i,j,l=1}^{n} \frac{\partial^3 f(\vec{0})}{\partial x_i \partial x_j \partial x_l} x_i x_j x_l$$