Lösungshinweise Blatt 12

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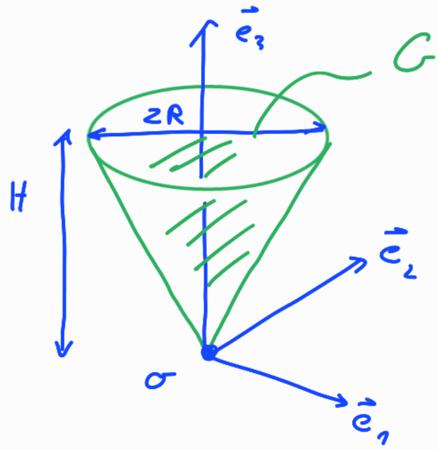

G : Vollkegel, Höhe H, Radius R

$$\frac{\partial \vec{G}}{\partial \rho} = \begin{pmatrix} u\cos\varphi \\ u\sin\varphi \\ H/R \end{pmatrix}, \quad \frac{\partial \vec{G}}{\partial \varphi} = \begin{pmatrix} -u\rho\sin\varphi \\ u\rho\cos\varphi \\ 0 \end{pmatrix},$$

$$\frac{\partial \vec{G}}{\partial u} = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ 0 \end{pmatrix};$$

$$\rightarrow \left|\det\left(\frac{\partial \vec{G}}{\partial \rho}, \frac{\partial \vec{G}}{\partial \varphi}, \frac{\partial \vec{G}}{\partial u}\right)\right| = \left|\begin{pmatrix} u\cos\varphi & -u\rho\sin\varphi & \rho\cos\varphi \\ u\sin\varphi & u\rho\cos\varphi & \rho\sin\varphi \\ H/R & 0 & 0 \end{pmatrix}\right|$$

$$= \left|-u\rho^2\sin^2\varphi \cdot H/R - u\rho^2\cos^2\varphi\, H/R\right| = Hu\rho^2/R$$

$$\rightarrow \underline{V(G)} = \int_G dV = \int_0^R \int_0^{2\pi} \int_0^1 \frac{H}{R}\rho^2 u \, d\rho \, d\varphi \, du$$

$$= \frac{H}{R} \underbrace{\int_0^R \rho^2 \, d\rho}_{= R^3/3} \underbrace{\int_0^{2\pi} d\varphi}_{= 2\pi} \underbrace{\int_0^1 u \, du}_{= 1/2} = \underline{\underline{\frac{1}{3}\pi R^2 H}}$$

54) $\vec{z}(\rho,\varphi,z) = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ z \end{pmatrix}$

$$\Rightarrow \det\left(\frac{\partial \vec{z}}{\partial \rho}, \frac{\partial \vec{z}}{\partial \varphi}, \frac{\partial \vec{z}}{\partial z}\right) = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \underline{\underline{\rho}}$$

$$\rightsquigarrow \int_Z f\,dV = \int_0^R \int_0^{2\pi} \int_0^H f(\varrho, \cancel{\varphi}, \cancel{z})\, \varrho \, d\varrho\, d\varphi\, dz$$

$$= \int_0^R f(\varrho)\varrho\, d\varrho \cdot \underbrace{\int_0^{2\pi} d\varphi}_{=2\pi} \cdot \underbrace{\int_0^H dz}_{= H}$$

$$= H \int_0^R f(\varrho)\, 2\pi\varrho \, d\varrho .$$

b) Kugelkoordinaten: $\vec{K}(r, \vartheta, \varphi) = r \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$.

$$\Rightarrow \det\left(\frac{\partial \vec{K}}{\partial r}, \frac{\partial \vec{K}}{\partial \vartheta}, \frac{\partial \vec{K}}{\partial \varphi}\right) = r^2 \sin\vartheta$$

↑ vgl. Vorl. $\S 8$.

$$\Rightarrow \int_K g\,dV = \int_0^R \int_0^{\pi} \int_0^{2\pi} g(r, \cancel{\vartheta}, \cancel{\varphi})\, r^2 \sin\vartheta \, dr\, d\vartheta\, d\varphi$$

$$= \int_0^R g(r)\, r^2\, dr \underbrace{\int_0^{\pi} \sin\vartheta\, d\vartheta}_{= -\cos\vartheta\big|_0^{\pi} = 2} \underbrace{\int_0^{2\pi} d\varphi}_{= 2\pi}$$

$$= \int_0^R g(r)\, 4\pi r^2\, dr$$

54 c) $I_3 = \int_Z m \varrho^2 \, dV \overset{a)}{=} H \int_0^R m \, 2\pi \varrho^3 \, d\varrho = H m \pi R^4/2$

mit $m = M/\pi R^2 H$ also $I_3 = \frac{M}{2} R^2$

54 d)

$$Q_{\text{Hülle}} = \lim_{R\to\infty} \int_{K_R} \varrho_e \, dV \overset{b)}{=} \int_0^\infty \varrho_e(r) \, 4\pi r^2 \, dr$$

$$= -\frac{4e}{a_0^3} \int_0^\infty r^2 e^{-2r/a_0} \, dr \underset{r = a_0 x/2}{=} -\frac{4e}{a_0^3} \left(\frac{a_0}{2}\right)^3 \underbrace{\int_0^\infty x^2 e^{-x} \, dx}_{= 2}$$

$$= -e .$$

55)

$$\text{div}\, \vec{A}(\vec{r}) = \text{div}\, \frac{\vec{r}}{r} = \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left(\frac{x_i}{r}\right) = \sum_i \left(\frac{1}{r} + x_i \frac{\partial}{\partial x_i} \frac{1}{r}\right)$$

$$= \sum_i \left(\frac{1}{r} - \frac{x_i}{r^2} \underbrace{\frac{\partial r}{\partial x_i}}_{= x_i/r}\right) = \sum_{i=1}^3 \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) = 2/r .$$

$$\text{div}\, \vec{a} \times \vec{r} = \underbrace{\frac{\partial}{\partial x_1}(a_2 x_3 - a_3 x_2)}_{=0} + \underbrace{\frac{\partial}{\partial x_2}(a_3 x_1 - a_1 x_3)}_{=0} + \underbrace{\frac{\partial}{\partial x_3}(a_1 x_2 - a_2 x_1)}_{=0}$$

$$= 0$$

$$\operatorname{div}\left(\langle \vec{a}, \vec{r}\rangle \vec{b}\right) = \frac{\partial}{\partial x_1}\left(a_1 x_1 + a_2 x_2 + a_3 x_3\right) b_1 + \frac{\partial}{\partial x_2}\left(a_1 x_1 + a_2 x_2 + a_3 x_3\right) b_2 + \frac{\partial}{\partial x_3}\left(a_1 x_1 + a_2 x_2 + a_3 x_3\right) b_3$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3 = \langle \vec{a}, \vec{b} \rangle .$$

56)

$$\operatorname{div} \vec{A} = \operatorname{div} c\vec{r} = 3c ,$$

$$\int_P \operatorname{div} \vec{A}\, dV = 3c \int_P dV = 3c \underbrace{\operatorname{Vol}(P)}_{= Ha^2/3} = Ha^2 c ,$$

$$\int_{\partial P} \vec{A}\, d\vec{f} \underset{\text{S.v.G.}}{=} \int_P \operatorname{div} \vec{A}\, dV = Ha^2/3 .$$