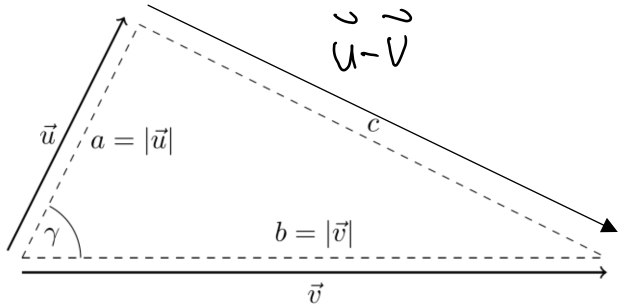
Lösungshinweise Blatt 2

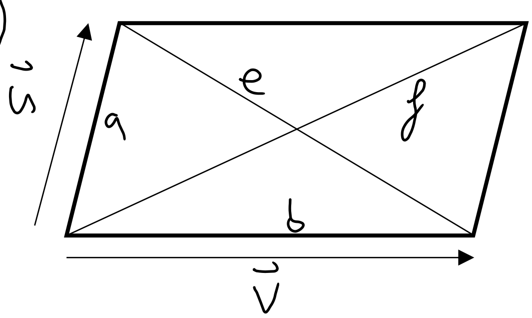
6. a) Kosinussatz:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$



b) $c^2 = \langle \vec u - \vec v, \vec u - \vec v\rangle = \underbrace{\langle \vec u, \vec u\rangle}_{=a^2} + \underbrace{\langle \vec v, \vec v\rangle}_{=b^2} - 2\underbrace{\langle \vec u, \vec v\rangle}_{ab\cos\gamma}$

c)



$e = |\vec v - \vec u|$, $f = |\vec v + \vec u|$

$$\begin{aligned} e^2 + f^2 &= \langle \vec v - \vec u, \vec v - \vec u\rangle + \langle \vec v + \vec u, \vec v + \vec u\rangle \\ &= 2|\vec v|^2 + 2|\vec u|^2 - \cancel{2\langle \vec v, \vec u\rangle} + \cancel{2\langle \vec v, \vec u\rangle} \\ &= 2(a^2 + b^2) \end{aligned}$$

d) zu zeigen: $(\vec v - \vec u) \perp (\vec v + \vec u)$

$$\langle \vec v - \vec u, \vec v + \vec u\rangle = |\vec v|^2 - |\vec u|^2 = a^2 - b^2 \overset{a=b}{=} 0 \ \checkmark$$

7. a) $\vec u = u_1\vec e_1 + u_2\vec e_2 + u_3\vec e_3$

b) (i) z.B. für u_1:

$$\langle \vec e_1, \vec u\rangle = u_1\underbrace{\langle \vec e_1, \vec e_1\rangle}_{=1} + u_2\underbrace{\langle \vec e_1, \vec e_2\rangle}_{=0} + u_3\underbrace{\langle \vec e_1, \vec e_3\rangle}_{=0} = u_1$$

analog für u_2 & u_3.

(ii) $\vec u = \sum_{i=1}^{3} u_i \vec e_i$, $\vec v = \sum_{i=1}^{3} v_i \vec e_i$

$$\rightarrow \langle \vec u, \vec v\rangle = \sum_{i=1}^{3}\sum_{j=1}^{3} u_i v_j \underbrace{\langle \vec e_i, \vec e_j\rangle}_{=\delta_{ij}} = \sum_{i=1}^{3} u_i v_i$$

$=\delta_{ij}$, alle Terme in der Summe mit $i \neq j$ verschwinden!

(iii) $|\vec{u}| = \sqrt{\langle \vec{u}, \vec{u} \rangle} \overset{(ii)}{=} \sqrt{\sum_{i=1}^{3} u_i^2}$

c)

- $|\vec{f}_1|^2 = \frac{1}{2} \langle \vec{e}_1 + \vec{e}_2, \vec{e}_1 + \vec{e}_2 \rangle = \frac{1}{2} \left(\underbrace{\langle \vec{e}_1, \vec{e}_1 \rangle}_{=1} + \underbrace{\langle \vec{e}_2, \vec{e}_2 \rangle}_{=1} + 2 \underbrace{\langle \vec{e}_1, \vec{e}_2 \rangle}_{=0} \right) = 1$
- $|\vec{f}_2|^2 = \frac{1}{2} \left(\langle \vec{e}_1, \vec{e}_1 \rangle + \langle \vec{e}_2, \vec{e}_2 \rangle - 2 \langle \vec{e}_1, \vec{e}_2 \rangle \right) = 1$
- $|\vec{f}_3|^2 = |\vec{e}_3|^2 = 1$

- $\langle \vec{f}_1, \vec{f}_2 \rangle = \frac{1}{2} \left(\overbrace{\langle \vec{e}_1, \vec{e}_1 \rangle}^{=1} - \overbrace{\langle \vec{e}_2, \vec{e}_2 \rangle}^{=1} \right) = 0$
- $\langle \vec{f}_1, \vec{f}_3 \rangle = \frac{1}{\sqrt{2}} \left(\underbrace{\langle \vec{e}_1, \vec{e}_3 \rangle}_{=0} + \underbrace{\langle \vec{e}_2, \vec{e}_3 \rangle}_{=0} \right) = 0$
- $\langle \vec{f}_2, \vec{f}_3 \rangle = \frac{1}{\sqrt{2}} \left(\langle \vec{e}_1, \vec{e}_3 \rangle - \langle \vec{e}_2, \vec{e}_3 \rangle \right) = 0$

$\longrightarrow \langle \vec{f}_i, \vec{f}_j \rangle = \delta_{ij}$, B' ist eine ONB.

d) Die Komponenten eines Vektors $\vec{a}$ in der Basis B' sind gegeben durch $\langle \vec{f}_i, \vec{a} \rangle = a_i'$.

- $u_1' = \frac{1}{\sqrt{2}} \left(\langle \vec{e}_1, \vec{u} \rangle + \langle \vec{e}_2, \vec{u} \rangle \right) = \frac{1}{\sqrt{2}} (u_1 + u_2) = \frac{3}{\sqrt{2}}$
- $u_2' = \frac{1}{\sqrt{2}} \left(\langle \vec{e}_1, \vec{u} \rangle - \langle \vec{e}_2, \vec{u} \rangle \right) = \frac{1}{\sqrt{2}} (u_1 - u_2) = -\frac{1}{\sqrt{2}}$

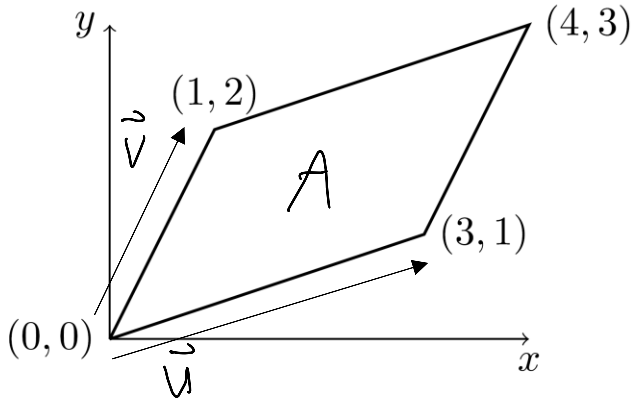
- $u_3' = \langle \vec{e}_3, \vec{u} \rangle = u_3 = 0$

$$\longrightarrow \vec{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}_{B'}$$

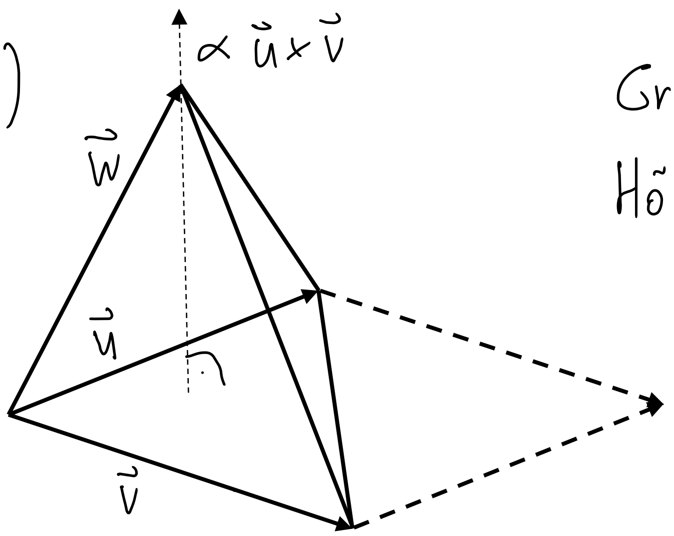
- $V_1' = \frac{1}{\sqrt{2}}(V_1 + V_2) = -\frac{1}{\sqrt{2}}$
- $V_2' = \frac{1}{\sqrt{2}}(V_1 - V_2) = \frac{5}{\sqrt{2}}$ $\longrightarrow \vec{v} = \begin{pmatrix} -1/\sqrt{2} \\ 5/\sqrt{2} \\ 4 \end{pmatrix}_{B'}$
- $V_3' = V_3 = 4$

8. a) $A = |\vec{u} \times \vec{v}| = \left| \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right|$

$= \left| \begin{pmatrix} 0 \\ 0 \\ 3\cdot 2 - 1\cdot 1 \end{pmatrix} \right| = 5$



b)



Grundfläche: $\frac{1}{2}|\vec{u} \times \vec{v}|$

Höhe: $|\langle \vec{u} \times \vec{v}, \vec{w} \rangle| / |\vec{u} \times \vec{v}|$

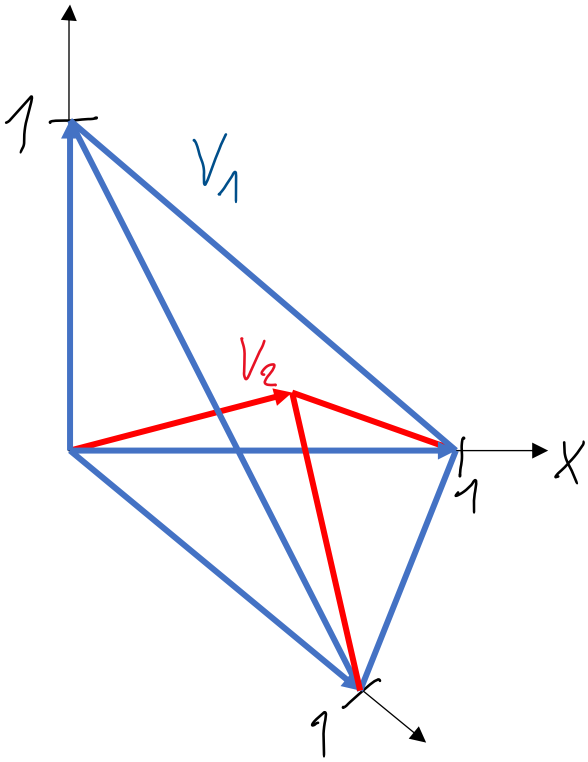
($\hat{=}$ Komponente von $\vec{w}$ entlang $\vec{u} \times \vec{v}$)

$\longrightarrow V = \frac{1}{3} \cdot \frac{1}{2} \cdot |\vec{u} \times \vec{v}| \cdot \frac{|\langle \vec{u} \times \vec{v}, \vec{w} \rangle|}{|\vec{u} \times \vec{v}|}$

$= \frac{1}{6} |\langle \vec{u} \times \vec{v}, \vec{w} \rangle|$

c) $V_1 = \frac{1}{6} |\langle \underbrace{\vec{e}_1 \times \vec{e}_2}_{\vec{e}_3}, \vec{e}_3 \rangle| = \frac{1}{6}$

$V_2 = \frac{1}{6} |\langle \vec{e}_1 \times \vec{e}_2, \frac{1}{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) \rangle| = \frac{1}{18} = V_1/3$



d) Das Volumen des von $\vec{u}, \vec{v}$ & $\vec{w}$ aufgespannten Tetraeders ändert sich nicht wenn man die Vektoren durchtauscht.