Lösungshinweise Blatt 4

16)

$$\left(\tfrac{1}{x^n}\right)' = \frac{1}{h}\left(\frac{1}{(x+h)^n} - \frac{1}{x^n}\right)$$

$$= \frac{1}{h} \frac{x^n - (x+h)^n}{(x+h)^n x^n} = \frac{1}{h}\left(\frac{\cancel{x^n} - \cancel{x^n} - n h x^{n-1} - \mathcal{O}(h^2)}{(x+h)^n x^n}\right)$$

$$= -n \frac{1}{(x+h)^n x} - \mathcal{O}(h) \xrightarrow[h=0]{} -\frac{n}{x^{n+1}}$$

17) a)

$$\left((1-x)^{-1/2}\right)' = \frac{1}{2}(1-x)^{-3/2} \xrightarrow[x=0]{} \frac{1}{2}$$

$$\left((1-x)^{-1/2}\right)'' = \frac{3}{4}(1-x)^{-5/4} \longrightarrow \frac{3}{4}$$

→ Taylor-Entw. 2 Ord. :

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2$$

b)

$$E = \frac{1}{\sqrt{\underbrace{1-v^2/c^2}_{x \ll 1}}} m c^2 \overset{a)}{=} \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\cancel{\frac{v^4}{c^4}}\right) m c^2$$

$$= m c^2 + \frac{m}{2}\frac{v^2}{c^2}$$

bis auf Konstante mc^2 ist für $v \ll c$

rel. Energie E gleich klassischen kin. Energie.

c) $$\frac{(E - mc^2) - \frac{1}{2} m v^2}{\frac{1}{2} m v^2}$$

a) $$= \frac{\frac{1}{2} v^2 + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 c^2 - \frac{1}{2} v^2}{\frac{1}{2} v^2}$$

$$= \frac{3}{4} \underbrace{v^2/c^2}_{= 1/100} = 0.0075 \mathrel{\hat{=}} \underline{\underline{0{,}75\,\%}}$$

18) a) $\ln(1+x) \simeq x$

b) $$\left(\ln(1-x)\right)' = -\frac{1}{1-x} \xrightarrow{x=0} -1$$

$$\left(\ln(1-x)\right)'' = -\frac{1}{(1-x)^2} \longrightarrow -1$$

$$\left(\ln(1-x)\right)''' = -\frac{2}{(1-x)^3} \longrightarrow -2$$

$$\left(\ln(1-x)\right)^{(4)} = -\frac{2\cdot 3}{(1-x)^4} \longrightarrow -2\cdot 3$$

$$\left(\ln(1-x)\right)^{(5)} = -\frac{2\cdot 3\cdot 4}{(1-x)^5} \longrightarrow -2\cdot 3\cdot 4$$

$$\vdots$$

$$\left(\ln(1-x)\right)^{(n)} = -\frac{(n-1)!}{(1-x)^n} \longrightarrow -(n-1)!$$

Taylor

$$\rightarrow \quad \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

1)

$$\left(1+\alpha/n\right)^n = e^{n \ln(1+\alpha/n)} \simeq e^{\alpha}.$$

$$\frac{1}{n} \ll 1: \ln\left(1+\frac{\alpha}{n}\right) \overset{(a)}{\simeq} \frac{\alpha}{n}$$

19)

$$(\sin x)^{(0)} = \sin x$$

$$(\sin x)^{(1)} = \cos x$$

$$(\sin x)^{(2)} = -\sin x$$

$$(\sin x)^{(3)} = -\cos x$$

$$(\sin x)^{(4)} = \sin x$$

$$\rightsquigarrow (\sin x)^{(2\ell)} = (-1)^{\ell} \sin x$$

$$(\sin x)^{(2\ell+1)} = (-1)^{\ell} \cos x$$

$$(\cos x)^{(0)} = \cos x$$

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$$\rightsquigarrow (\cos x)^{(2\ell)} = (-1)^{\ell} \cos x$$

$$(\cos x)^{(2\ell+1)} = -(-1)^{\ell} \sin x$$

20) a)

$$e^x \underset{\text{Taylor}}{=} \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\overbrace{(e^x)^{(n)}}^{e^x}\Big|_{x=0}}_{= e^0 = 1} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

b)

$$\sin x \underset{\text{Taylor}}{=} \sum_{n=0}^{\infty} \frac{1}{n!} (\sin x)^{(n)}\Big|_{x=0} x^n$$

$$= \sum_{l=0}^{\infty} \left\{ \frac{1}{(2l)!} (\sin x)^{(2l)}\Big|_{x=0} x^{2l} + \frac{1}{(2l+1)!} (\sin x)^{(2l+1)}\Big|_{x=0} x^{2l+1} \right\}$$

$$\overset{19}{=} \sum_{l=0}^{\infty} (-1)^l \frac{x^{2l+1}}{(2l+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \dots$$

$$\cos x \underset{\text{Taylor}}{=} \sum_{n=0}^{\infty} \frac{1}{n!} (\cos x)^{(n)}\Big|_{x=0} x^n$$

$$= \sum_{l=0}^{\infty} \left\{ \frac{1}{(2l)!} (\cos x)^{(2l)}\Big|_{x=0} x^{2l} + \frac{1}{(2l+1)} (\cos x)^{(2l+1)}\Big|_{x=0} x^{2l+1} \right\}$$

$$\overset{19}{=} \sum_{l=0}^{\infty} (-1)^l \frac{x^{2l}}{(2l)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots - \dots$$

2 1)

- $\frac{\partial f}{\partial x_1} = -\sin x_1 \sin x_2 \quad , \quad \frac{\partial f}{\partial x_2} = \cos x_1 \cos x_2$

- $\frac{\partial g}{\partial x_1} = -2\lambda x_1 e^{-\lambda x_1^2} \sin(k x_2)$

  $\frac{\partial g}{\partial x_2} = k e^{-\lambda x_1^2} \cos(k x_2)$

- $h(\vec{x}) = \sum_i a_i x_i \rightsquigarrow \frac{\partial h}{\partial x_\ell} = a_\ell$

- $j(\vec{x}) = \exp(-\mu \sum_i x_i^2)$

$\rightarrow \quad \frac{\partial j(\vec{x})}{\partial x_\ell} = -2\mu x_\ell \, j(\vec{x})$