Lösungshinweise Blatt 8

41) • $y' = y/(1+x)^2$: lineare, homogene DGL

$\rightarrow \quad y(x) = y_0\, e^{F(x)}$ mit Stammfkt.

$$F(x) = \int_0^x \frac{du}{(1+u)^2} = -\frac{1}{1+u}\Big|_0^x = 1 - \frac{1}{1+x} = \frac{x}{1+x}$$

d.h. $\quad y(x) = y_0\, e^{x/1+x}$.

• $y' = x^3$: triviale DGL :

$$y(x) = \int_0^x u^3\,du + y_0 = \frac{x^4}{4} + y_0 \quad .$$

• $y' = y^3$: separierbare DGL :

T. d. V. : $\displaystyle \int_{y_0}^{y_x} \frac{dy}{y^3} = \int_0^x 1\,du = x$

$$\underbrace{\int_{y_0}^{y_x} \frac{dy}{y^3}} = -\frac{1}{2y^2}\Big|_{y_0}^{y_x} = \frac{1}{2y_0^{\,2}} - \frac{1}{2y_x^{\,2}}$$

$$\rightarrow \quad y(x) = \frac{1}{\sqrt{\frac{1}{y_0^2} - 2x}}$$

⌈ Test: $y(0) = y_0$ ✓

$$y'(x) = \frac{-1}{2\sqrt{\frac{1}{y_0^2}-2x}^{\,3}}\,(-2) = y(x)^3 \quad ✓$$ ⌟

• $y' = \sqrt{y}/1+x$: separierbare DGL, $\longrightarrow$

T. d. V.:

$$\underbrace{\int_{y_0}^{y_x} \frac{dy}{\sqrt{y}}}_{=\; 2\sqrt{y}\,\Big|_{y_0}^{y_x} \;=\; 2\sqrt{y_x} - 2\sqrt{y_0}} \overset{!}{=} \int_0^x \frac{du}{1+u} = \ln(1+x)$$

$$\rightarrow \quad y(x) = \left(\frac{1}{2} \ln(1+x) + \sqrt{y_0} \right)^2$$

⌜ Test: $y(0) = y_0$ ✓

$$y'(x) = \left(\ln(1+x) + 2\sqrt{y_0} \right) \cdot \frac{1}{2} \cdot \frac{1}{1+x}$$

$$= \frac{\frac{1}{2}\ln(1+x) + \sqrt{y_0}}{1+x} = \frac{\sqrt{y(x)}}{1+x} \quad ✓$$ ⌟

42 a) $0 \overset{!}{=} g - \alpha v_g^2 \rightarrow v_g = \sqrt{g/\alpha}$

b) $\dot{v} = g - \alpha v^2 = g\left(1 - \left(\sqrt{\frac{\alpha}{g}}\, v\right)^2\right)$

d. h. $\dot{v} = g\left(1 - \left(v/v_g\right)\right)^2$

↗ separierbare DGL, T. d. V.:

$$\int_0^{v_t} \frac{dv}{1 - (v/v_g)^2} = g \int_0^t dt' = g\,t$$

$$g t = \int_0^{v_t} \frac{dv}{1-(v/v_g)^2} = v_g \int_0^{v_t/v_g} \frac{dx}{1-x^2}$$

$v = v_g x$

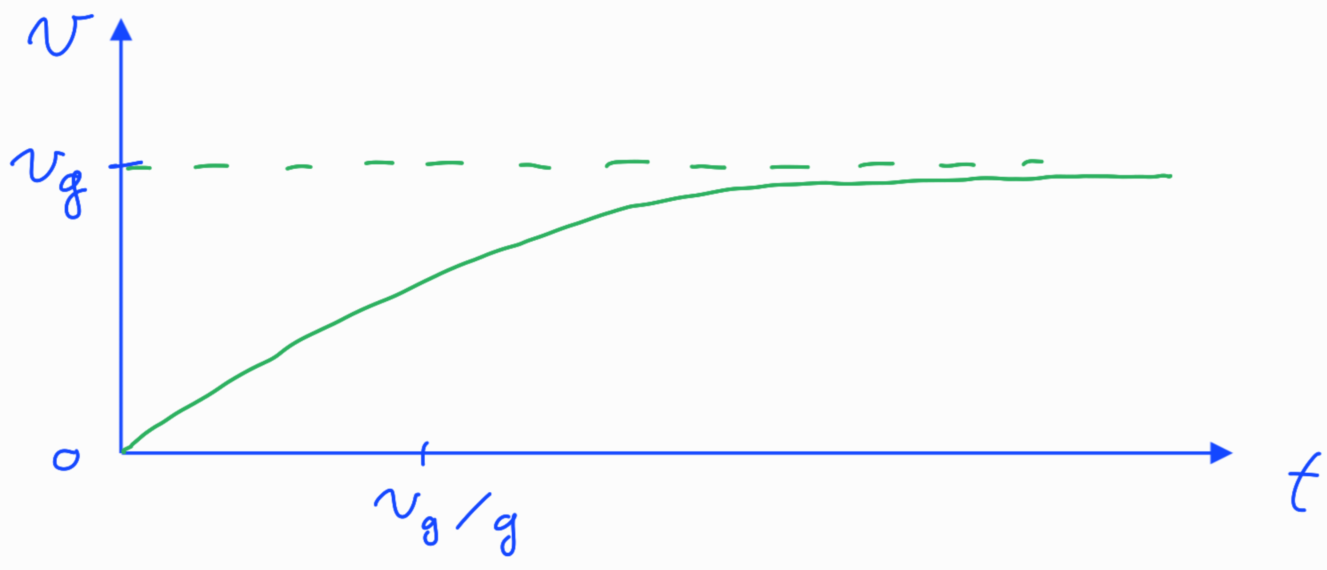
$$= \frac{v_g}{2} \ln\left(\frac{1+x}{1-x}\right)\Big|_0^{v_t/v_g} = \frac{v_g}{2} \ln\left(\frac{1+v_t/v_g}{1-v_t/v_g}\right)$$

d.h. $$e^{2gt/v_g} = \frac{1+v_t/v_g}{1-v_t/v_g}$$

auflösen nach v_t ergibt nach einfacher Rechnung

$$\boxed{v(t) = \frac{e^{2gt/v_g}-1}{e^{2gt/v_g}+1} \cdot v_g}$$

$$\left[= v_g \frac{e^{gt/v_g} - e^{-gt/v_g}}{e^{gt/v_g} + e^{gt/v_g}} v_g = v_g \tanh \frac{gt}{v_g} \right]$$



c) $v\left(\frac{v_g}{g}\right) = v_g \frac{e^2-1}{e^2+1} \simeq 0.76\, v_g$

$\rightarrow$ nach $T = \frac{v_g}{g} = \frac{56\,m/s}{10\,m/s^2} \simeq 5{,}6\,s$

wird $v(T) = 0.76\, v_g \simeq 150\,km/h$

erreicht.

43 a) $u(t) = u_0 e^{\gamma t}$ ist Lsg. der DGL

$\dot{u} = \gamma u$ zum A.W. u_0 bei $t = 0$.

b)

- solange $u(t) \ll 1/\beta$ ist $1 - \beta u(t) \simeq 1$

und somit $\dot{u}(t) = \gamma\, u(t)$,

$\overset{a)}{\rightarrow}$ $u(t) = u_0\, e^{\gamma t}$.

- mit zunehmender Zeit flacht

Kurve ab, verschwindende Steigung

für $t \rightarrow \infty$; dann $0 \overset{!}{=} \gamma(\underbrace{1-\beta u_\infty}_{=0}) u_\infty$ $\neq 0$

$\rightarrow$ $u_\infty = 1/\beta$

c)

$$\dot{n}(t) = -\underbrace{\frac{n_\infty}{\left(1+\left(\frac{n_\infty}{n_0}-1\right)e^{-\gamma t}\right)^2}} \cdot \underbrace{\left(\frac{n_\infty}{n_0}-1\right)e^{-\gamma t}} \cdot (-\gamma)$$

$$= \gamma \frac{n(t)^2}{n_\infty} \left(\frac{n_\infty}{n(t)} - 1\right)$$

$$= \gamma\, n(t) \left(1 - \frac{n(t)}{n_\infty}\right) \quad ✓$$

$$n(0) = \frac{n_\infty}{1+\left(\frac{n_\infty}{n_0}-1\right)} = n_0 \quad ✓$$

- $t \ll \frac{1}{\gamma} \ln \frac{n_\infty}{n_0} \rightarrow e^{\gamma t} \ll \frac{n_\infty}{n_0}$

$$\rightarrow \frac{n_\infty}{n_0} e^{-\gamma t} \gg 1 \geq e^{-\gamma t}$$

$$\rightarrow n(t) \approx \frac{n_\infty}{\frac{n_\infty}{n_0} e^{-\gamma t}} = n_0 e^{-\gamma t}$$

- $t \gg 1/\gamma \ln \frac{n_\infty}{n_0} \rightarrow \left(\frac{n_\infty}{n_0} - 1\right) e^{-\gamma t} \ll 1$

$$\rightarrow n(t) \approx n_\infty$$

∘ $\gamma = 1, \quad u_0 = 0.05\, u_\infty$

$$\rightsquigarrow \quad \frac{1}{\gamma} \ln \frac{u_\infty}{u_0} = \ln 20 \simeq 3{,}00$$