Lösungshinweise Blatt 9

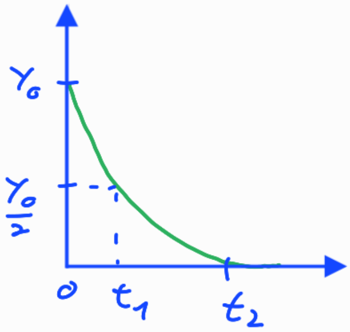
45 a) $\alpha y = I = -\frac{d}{dt}\left(\frac{1}{3}\pi y^3\right) = -\pi y^2 \dot{y}$

$\rightarrow \quad \dot{y} = -\frac{\alpha}{\pi}\frac{1}{y}$, Lsg. mittels

b) T. d. V.:

$$\underbrace{\int_{y_0}^{y_t} y\,dy}_{= \frac{1}{2}(y_t^2 - y_0^2)} \overset{!}{=} -\int_0^t \frac{\alpha}{\pi}\,dt = -\frac{\alpha}{\pi}t$$

d.h. $y(t) = \sqrt{y_0^2 - \frac{2\alpha}{\pi}t}$



c) Zur Zeit t_1 sei $y(t_1) = y_0/2$

$\rightarrow \quad \frac{2\alpha}{\pi}t_1 = \frac{3}{4}y_0^2$

$\rightarrow \quad y(t) = y_0\sqrt{1 - \frac{3}{4}t/t_1}$

$\rightarrow$ Trichter leer zur Zeit $t_2 = \frac{4}{3}t_1 = t_1 + \frac{1}{3}t_1$

= 1 Min + 20 Sek

(beachte: $V(y_0/2) = \frac{1}{8}V(y_0)$!)

46) a) Exponentialansatz: $x(t) = e^{\lambda t}$

Lsg $\iff \lambda^2 + \gamma\lambda - \frac{k}{m} = 0$

$\rightarrow \quad \lambda_{1/2} = -\frac{\gamma}{2} \pm \sqrt{\gamma^2/4 + k/m}$

mit $\beta := \sqrt{\gamma^2/4 + k/m}$ also $\lambda_{1/2} = -\frac{\gamma}{2} \pm \beta$

$\rightarrow$ spezielle Lsgen

$$x_1(t) = e^{-\gamma t/2}\, e^{+\beta t}$$

$$x_2(t) = e^{-\gamma t/2}\, e^{-\beta t}$$

$\rightarrow$ allg. Lsg.: $x(t) = e^{-\gamma t/2}\left(c_1 e^{\beta t} + c_2 e^{-\beta t}\right)$

b) Anfangsort x_0 und Anfangsgeschw. $v_0 = 0$

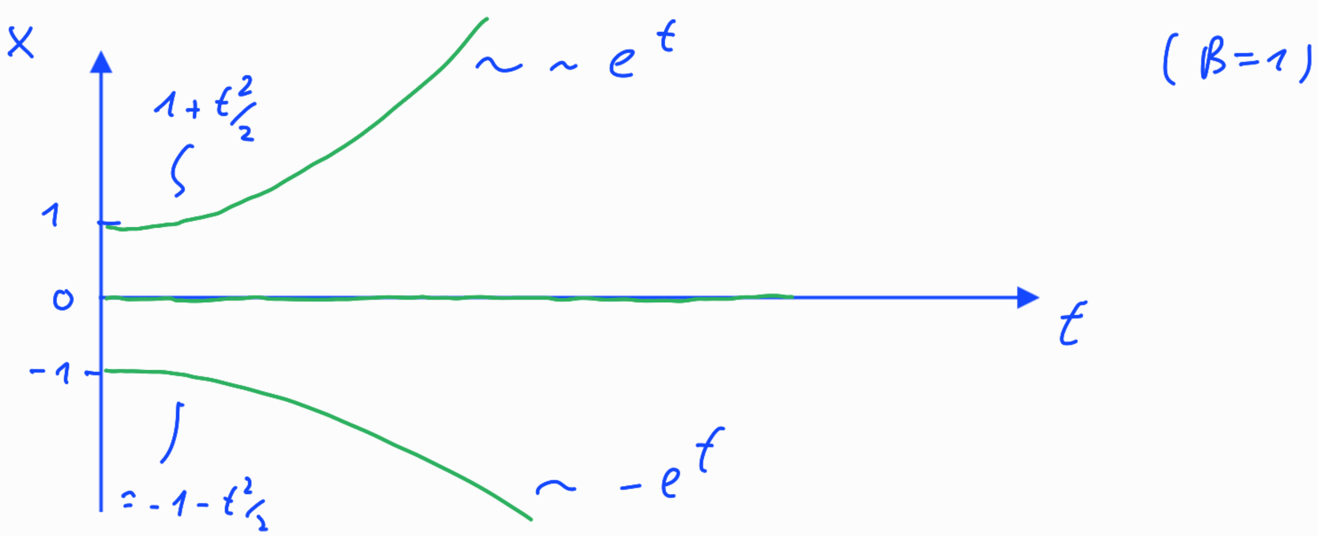
bei $t = 0$: $(\gamma = 0!)$

$$\begin{cases} x(0) = c_1 + c_2 = x_0 \\ \dot{x}(0) = \beta c_1 - \beta c_2 = 0 \end{cases}$$

$$\rightarrow c_1 = c_2 = x_0/2$$

u. l. $x(t) = \frac{x_0}{2}\left(e^{\beta t} + e^{-\beta t}\right)$

$$= x_0\left(1 + \tfrac{1}{2}\beta^2 t^2 + \mathcal{O}(\beta t)^4\right)$$



47 a)

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ b+a \end{pmatrix} = (a+b)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a-b \\ b-a \end{pmatrix} = (a-b)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

b)

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \vec{e}_i = \lambda_i \vec{e}_i$$

c)

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\neq$ (!)

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

d)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{pmatrix} = \begin{pmatrix} 10 & -12 \\ 19 & -24 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 13 & 14 & 15 \\ -19 & -20 & -21 \end{pmatrix}$$

48 a)

$$D_\varphi \vec{x} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1\cos\varphi & -x_2\sin\varphi \\ x_1\sin\varphi & +x_2\cos\varphi \end{pmatrix}$$

$$\rightsquigarrow |D_\varphi \vec{x}|^2 = x_1^2\cos^2\varphi + x_2^2\sin^2\varphi - 2\cancel{x_1x_2\cos\varphi\sin\varphi}$$

$$+ x_1^2\sin^2\varphi + x_2^2\cos^2\varphi + 2\cancel{x_1x_2\cos\varphi\sin\varphi}$$

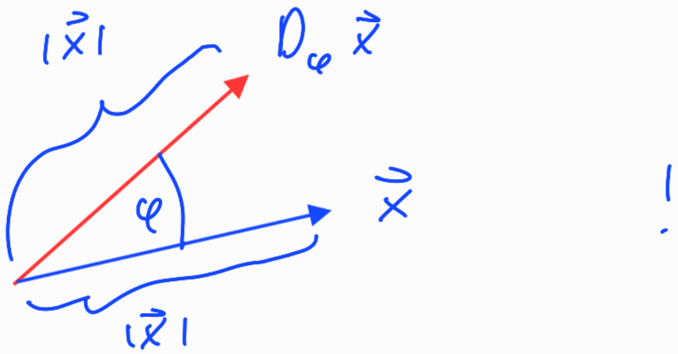
$$= x_1^2 + x_2^2 = |\vec{x}|^2$$

$$\frac{1}{|\vec{x}|\;|D_\varphi \vec{x}|} \langle \vec{x}, D_\varphi \vec{x} \rangle =$$

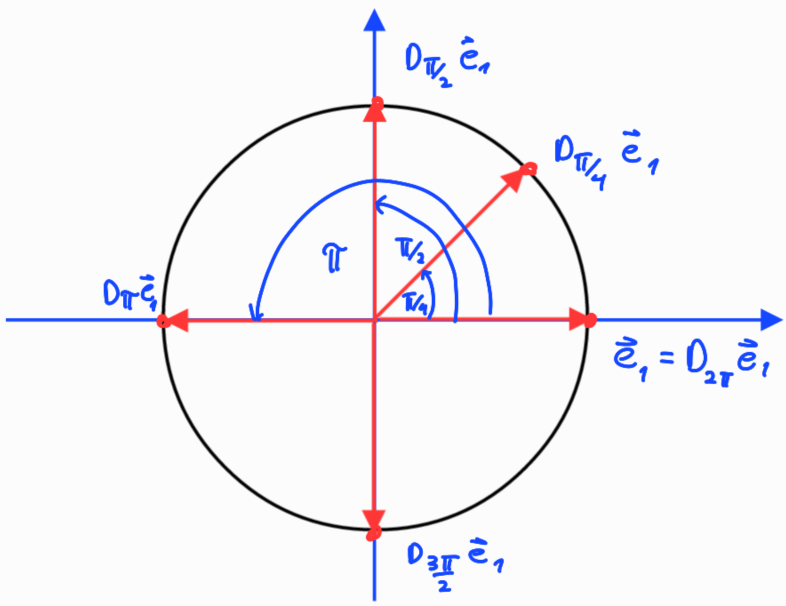
$$= \frac{1}{|x|^2} \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \cos\varphi - x_2 \sin\varphi \\ x_1 \sin\varphi + x_2 \cos\varphi \end{pmatrix} \right\rangle$$

$$= \frac{1}{|x|^2} \left(x_1^2 \cos\varphi - \cancel{x_1 x_2 \sin\varphi} + \cancel{x_2 x_1 \sin\varphi} + x_2^2 \cos\varphi \right)$$

$$= \cos\varphi \quad ! \qquad \text{d.h.} \quad \sphericalangle(\vec{x}, D_\varphi \vec{x}) = \varphi$$



b)



1)
$$D_\varphi D_\vartheta = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$$

$$= \begin{pmatrix} \overbrace{\cos\varphi\cos\vartheta - \sin\varphi\sin\vartheta}^{\cos(\varphi+\vartheta)} & \overbrace{-\cos\varphi\sin\vartheta - \sin\varphi\cos\vartheta}^{-\sin(\varphi+\vartheta)} \\ \underbrace{\sin\varphi\cos\vartheta + \cos\varphi\sin\vartheta}_{\sin(\varphi+\vartheta)} & \underbrace{-\sin\varphi\sin\vartheta + \cos\varphi\cos\vartheta}_{\cos(\varphi+\vartheta)} \end{pmatrix}$$

$$= D_{\varphi+\vartheta}$$

geometrisch: "Drehung um Winkel φ

+ Drehung um Winkel ϑ

= Drehung um Winkel $\varphi+\vartheta$"

komplexe Alternative:

- $z = x_1 + i x_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \hat{=} \vec{x}$

$$\rightarrow \underline{\underline{e^{i\varphi}}} z = (\cos\varphi + i\sin\varphi)(x_1 + i x_2)$$

$$= x_1\cos\varphi - x_2\sin\varphi + i(x_1\sin\varphi + x_2\cos\varphi)$$

$$= \begin{pmatrix} x_1\cos\varphi - x_2\sin\varphi \\ x_1\sin\varphi + x_2\cos\varphi \end{pmatrix} \hat{=} \underline{\underline{D_\varphi}}\,\vec{x} \quad !$$

$\rightarrow$ a) $\bullet\ |D_\varphi \vec{x}| = |e^{i\varphi} z| = \underbrace{|e^{i\varphi}|}_{=1} |z| = |z| = |\vec{x}|$ ✓

$\bullet\ \sphericalangle(\vec{x}, D_\varphi \vec{x}) = \underbrace{\arg(e^{i\varphi} z)}_{\parallel} - \arg(z)$

$= \overbrace{\varphi + \arg z} - \arg(z) = \varphi$ ✓

$\underbrace{\arg z - \arg(z)}_{=0}$

b) $D_{\pi/4}\, \vec{e}_1 = e^{i\pi/4} \cdot 1 = \frac{1+i}{\sqrt{2}}$

$D_{\pi/2}\, \vec{e}_1 = e^{i\pi/2} = i$

$D_{\pi}\, \vec{e}_1 = e^{i\pi} = -1$

$D_{3\pi/2}\, \vec{e}_1 = e^{i3\pi/2} = -i$

c)

$$D_\varphi D_\vartheta \vec{x} = e^{i\varphi} e^{i\vartheta} z = e^{i(\varphi+\vartheta)} z = D_{\varphi+\vartheta} \vec{x}$$