Nachtrag: Höhere partielle Ableitungen

- $$\frac{\partial^2 f}{\partial x_l^2} := \frac{\partial}{\partial x_l}\left(\frac{\partial f}{\partial x_l}\right)$$

- $$\frac{\partial^2 f}{\partial x_l \partial x_m} := \frac{\partial}{\partial x_l}\left(\frac{\partial f}{\partial x_m}\right)$$

Beispiel: $f(x_1, x_2) = x_1^2 (x_2+1)$

$$\rightarrow \quad \frac{\partial f}{\partial x_1} = 2x_1(x_2+1), \quad \frac{\partial f}{\partial x_2} = x_1^2$$

$$\rightarrow \quad \frac{\partial^2 f}{\partial x_1^2} = 2(x_2+1), \quad \frac{\partial^2 f}{\partial x_2^2} = 0$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 2x_1$$

$$= \ !$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 2x_1$$

S. v. Schwarz: $$\boxed{\frac{\partial^2 f}{\partial x_l \partial x_m} \overset{!}{=} \frac{\partial^2 f}{\partial x_m \partial x_l}}$$

Gradient:

$$\boxed{\operatorname{grad} f(\vec{x}) := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x})\, \vec{e}_i}$$

$$= \begin{pmatrix} \frac{\partial f(\vec{x})}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n}(\vec{x}) \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}}_{=:\, \vec{\nabla} \text{ "Nabla"}} f(\vec{x}) = \vec{\nabla} f(\vec{x})$$

- <u>Lineare Näherung</u>:

$$f(\vec{x} + \vec{h}) = f(\vec{x}) + \langle \operatorname{grad} f(\vec{x}), \vec{h} \rangle$$

- <u>Richtungsableitung</u>:

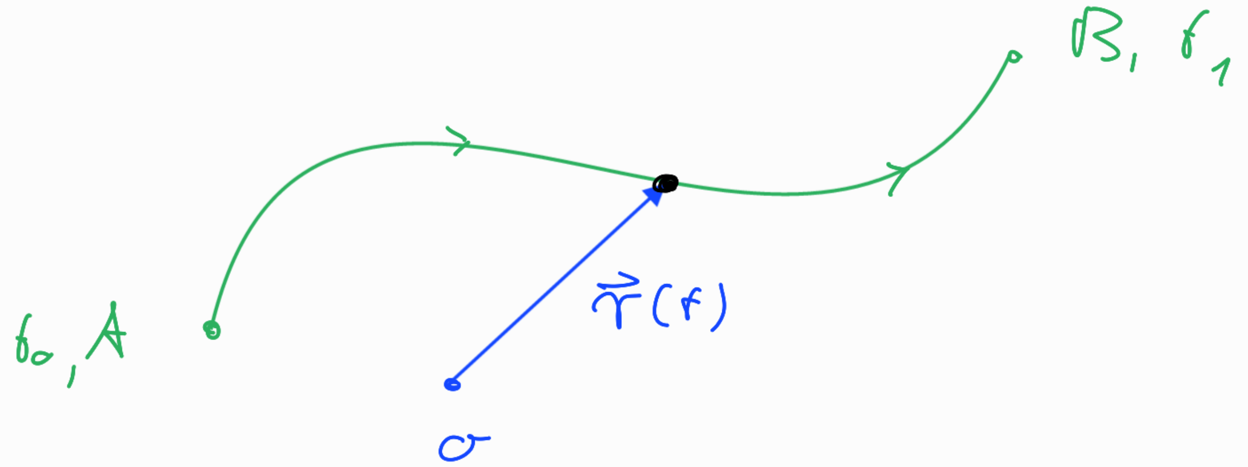
$$\partial_{\hat{u}} f(\vec{x}) = \langle \operatorname{grad} f(\vec{x}), \hat{u} \rangle$$

<u>Eigenschaften</u>:

- $\widehat{\operatorname{grad} f(\vec{x})}$ = Richtung $\hat{u}_0$ maximalen Anstiegs
- $|\operatorname{grad} f(\vec{x})|$ = Steigung in Richtung $\hat{u}_0$
- $\operatorname{grad} f(\vec{x}) \perp$ Hyperflächen "$f = c$".

- $\operatorname{grad} |\vec{r}| = \hat{r}$
- $\operatorname{grad} g(|\vec{r}|) = g'(|\vec{r}|)\, \hat{r}$.

<u>Kinematik</u>: Bahn, Geschwindigkeit, Beschl. eines Massenpunkts im Raum



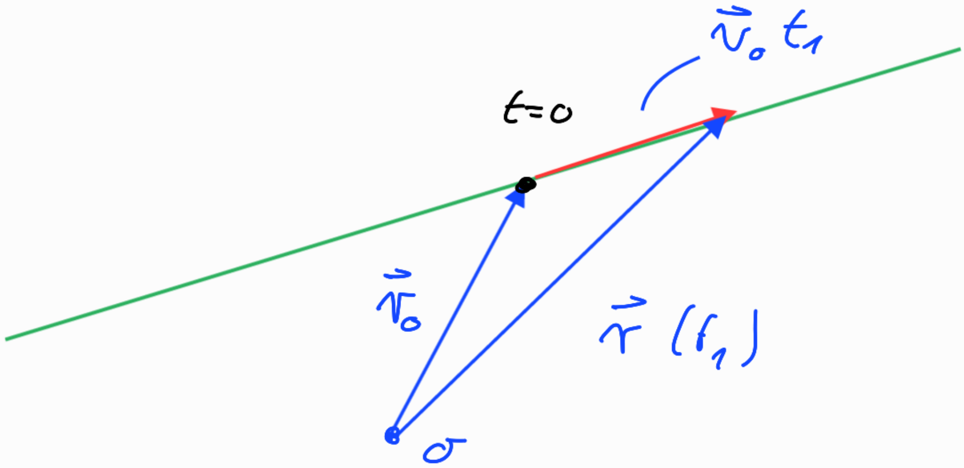
<u>Bahn</u> = Abb. $\vec{r} : [t_0, t_1] \longrightarrow \mathbb{R}^3$

$$t \longmapsto \vec{r}(t)$$

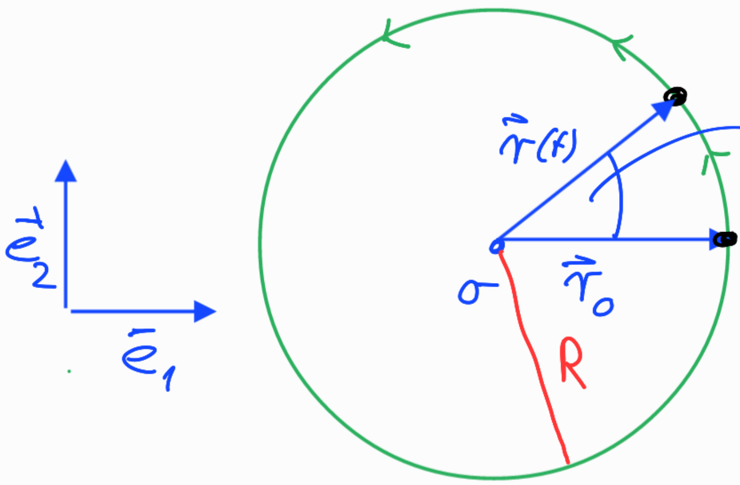
<u>Beispiele</u>: 1) geradlinig gleichförmige Bewegung

$\leadsto$

$$t \longmapsto \vec{r}(t) = \vec{r}_0 + \underline{\underline{\vec{v}_0}}\, t$$



<u>Bsp.:</u> <u>gleichförmige Kreisbewegung</u>



$\varphi(t) = \omega t$

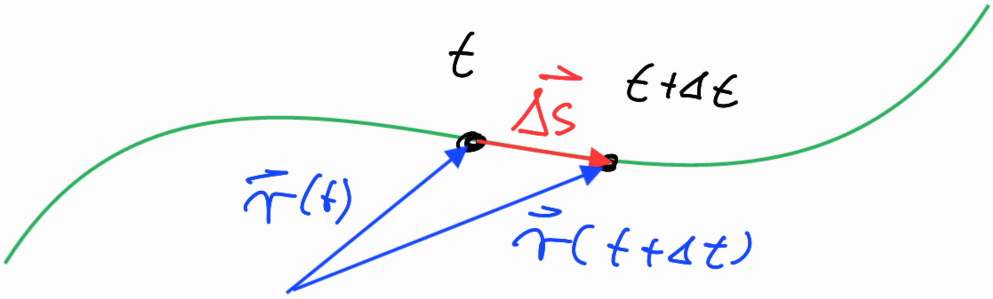
↑ Winkelgeschwindigkeit

$$\vec{r}(t) = R \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}$$

Umlaufzeit $T = \frac{2\pi}{\omega}$

$$\Rightarrow \vec{r}(t+T) \overset{!}{=} \vec{r}(t)$$

momentane Geschwindigkeit $\vec{v}(t)$:



$$\vec{v}(t) := \frac{\text{Ortsänderung}}{\text{Zeit}} = \frac{\vec{\Delta s}}{\Delta t}$$

$$\frac{\vec{\Delta s}}{\Delta t} = \frac{1}{\Delta t}\left(\vec{r}(t+\Delta t) - \vec{r}(t)\right)$$

$$\rightarrow \boxed{\vec{v}(t) = \dot{\vec{r}}(t)} := \lim_{h \to 0} \frac{1}{h}\left(\vec{r}(t+h) - \vec{r}(t)\right)$$

Vektorsubtr.

Skalarmultiplikation.

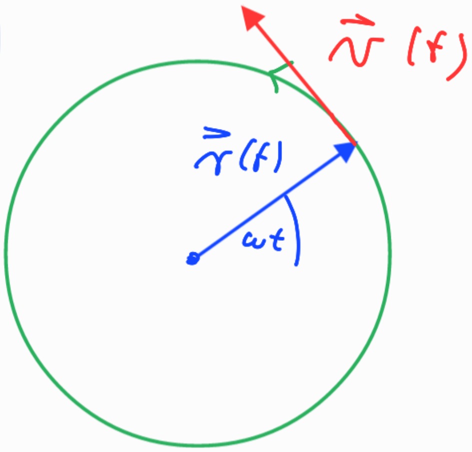
Beisp. 1: gerad. gl.f. Bewegung

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t$$

$$\rightarrow \vec{v}(t) = \dot{\vec{r}}(t) = \vec{v}_0 \quad (\text{konstant})$$

Bsp. 2) gleichf. Kreisbewegung:

$$\vec{r}(t) = R \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}$$

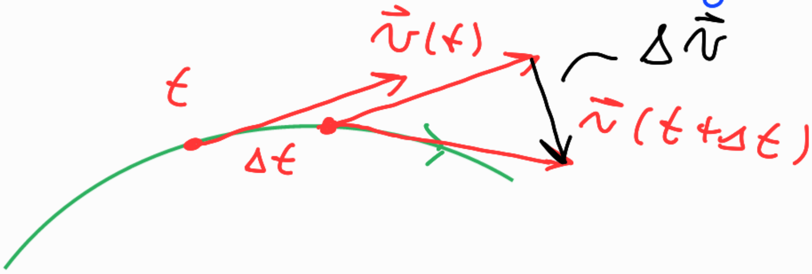


$$\vec{v}(t) = \dot{\vec{r}}(t) = R \begin{pmatrix} -\omega \sin \omega t \\ \omega \cos \omega t \end{pmatrix}$$

$$= \omega R \begin{pmatrix} -\sin \omega t \\ \cos \omega t \end{pmatrix}$$

$$|\vec{v}(t)| = \omega R$$

<u>momentane Beschleunigung</u> $\vec{a}(t)$



$$\vec{a}(t) = \frac{\text{Geschwindigkeitsänderung}}{\text{Zeit}} = \frac{\Delta \vec{v}}{\Delta t}$$

$$= \frac{1}{\Delta t} \left(\vec{v}(t+\Delta t) - \vec{v}(t) \right) \xrightarrow[\Delta t \to 0]{}$$

$$\rightarrow \quad \vec{a}(t) = \dot{\vec{v}}(t) = \frac{d}{dt}\vec{v}(t) = \frac{d^2}{dt^2}\vec{r}(t) = \ddot{\vec{r}}(t)$$

$$\boxed{\vec{a}(t) = \dot{\vec{v}}(t) = \ddot{\vec{r}}(t)}$$

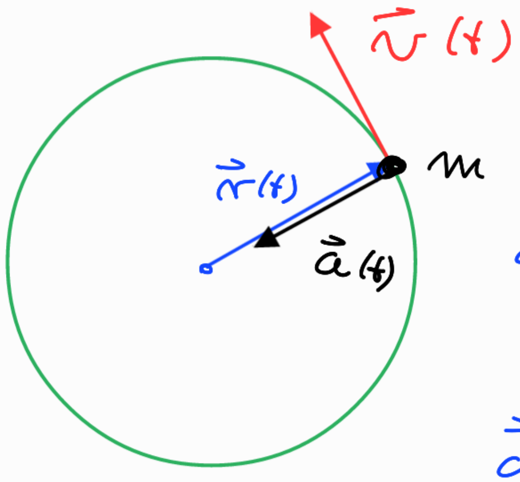
Bsp 1) gerade gl.f. Bewegung:

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t$$

$$\vec{v}(t) = \dot{\vec{r}}(t) = \vec{v}_0$$

$$\Rightarrow \vec{a}(t) = \dot{\vec{v}}(t) = \vec{0}$$

Bsp 2) gl.f. Kreisbewegung



$$\vec{r}(t) = R\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix}$$

$$\xrightarrow{\frac{d}{dt}} \quad \vec{v}(t) = \omega R\begin{pmatrix}-\sin\omega t\\ \cos\omega t\end{pmatrix}$$

$$\xrightarrow{\frac{d}{dt}} \quad \vec{a}(t) = \omega R\begin{pmatrix}-\omega\cos\omega t\\ -\omega\sin\omega t\end{pmatrix}$$

$$\underline{\vec{a}(t)} = -\omega^2 R\underbrace{\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix}}_{\vec{r}(t)} = -\omega^2 \underline{\vec{r}(t)}$$

$$|\vec{a}(t)| = \omega^2 R$$

Bsp. 3 : konstant beschleunigte Bewegung

d.h.: $\vec{a}(t) \overset{!}{=} \vec{g} \neq \vec{0}$ $\quad \downarrow \vec{g}$

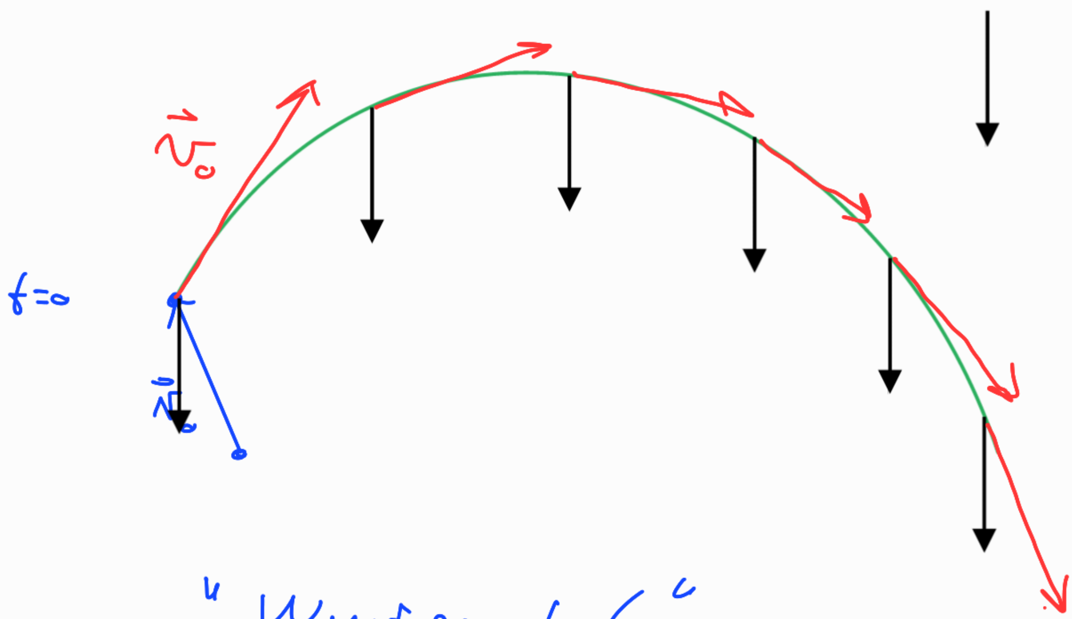
$$\boxed{\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{g}\, t^2}$$

$\xrightarrow{d/dt}$ $\vec{v}(t) = \vec{v}_0 + \vec{g}\, t$

$\xrightarrow{d/dt}$ $\vec{a}(t) = \vec{g}$ ✓

Anfangsort: $\vec{r}(0) = \vec{r}_0$

Anfangsgeschw.: $\vec{v}(0) = \vec{v}_0$



„Wurfparabel“

Ableitung vektorwertiger Fktn:

$$\vec{f} : D \longrightarrow \mathbb{R}^n \qquad D \subset \mathbb{R}$$

$$x \longmapsto \vec{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

$$\vec{f}'(x) = \lim_{h \to} \frac{1}{h}\left(\vec{f}(x+h) - \vec{f}(x)\right)$$

$$= \begin{pmatrix} f_1'(x) \\ f_2'(x) \\ \vdots \\ f_n'(x) \end{pmatrix} ,$$

partielle Ableitung einer vektorwertigen Fkt.:

$$\vec{A} : D \longrightarrow \mathbb{R}^n , \qquad D \subset \mathbb{R}^m$$

$$\vec{x} \longmapsto \vec{A}(\vec{x})$$

$$\longrightarrow \quad \frac{\partial \vec{A}}{\partial x_e}(\vec{x}) = \lim_{h \to 0} \frac{1}{h}\left(\vec{A}(\vec{x} + h\vec{e}_e) - \vec{A}(\vec{x})\right) .$$

Bsp.: lokale Basis in Polarkoordinaten:

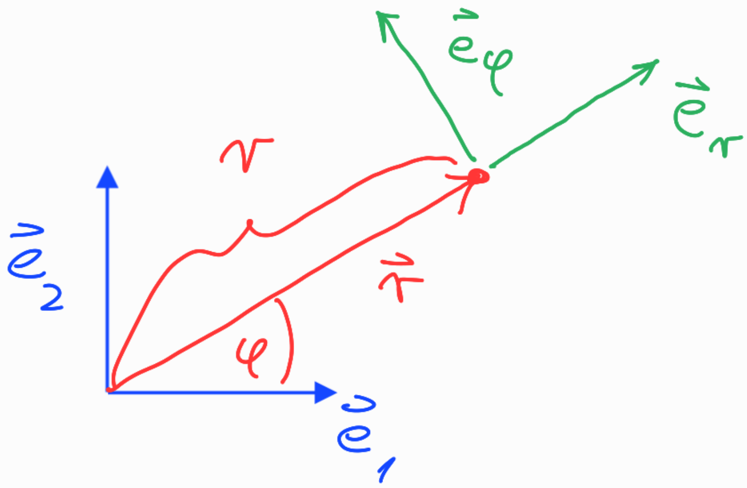


Abb.: $\vec{r}: \mathbb{R}_+ \times [0, 2\pi[\longrightarrow \mathbb{R}^2$

$$(r, \varphi) \longmapsto \vec{r}(r, \varphi) = r \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

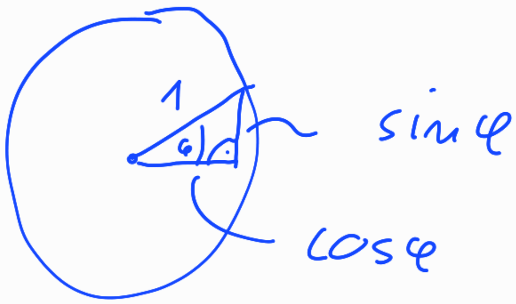
$$\rightarrow \quad \vec{e}_r(r, \varphi) := \frac{1}{\left| \frac{\partial \vec{r}}{\partial r} \right|} \frac{\partial \vec{r}}{\partial r}$$

$$\vec{e}_\varphi(r, \varphi) := \frac{1}{\left| \frac{\partial \vec{r}}{\partial \varphi} \right|} \frac{\partial \vec{r}}{\partial \varphi}$$

$$\Rightarrow: \quad \frac{\partial \vec{r}}{\partial r} = \frac{\partial}{\partial r}\left(r \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \right) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$\rightarrow \quad \vec{e}_r = \frac{1}{\underbrace{\left| \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \right|}_{=1}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$\left[\quad \left|\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\right| = \Big(\underbrace{\cos^2\varphi + \sin^2\varphi}_{\hat{=}\,1\,!}\Big)^{1/2} = 1\right.$$



$$\left.\right]$$

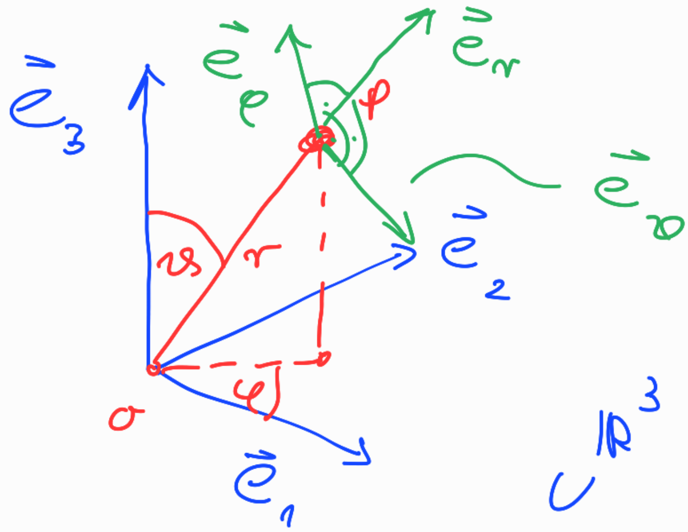
$$\vec{e}_\varphi: \qquad \frac{\partial\vec{r}}{\partial\varphi} = \frac{\partial}{\partial\varphi}\left(r\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\right)$$

$$= r\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix},$$

$$\longrightarrow \quad \left|\frac{\partial\vec{r}}{\partial\varphi}\right| = r$$

$$\vec{e}_\varphi = \frac{1}{\cancel{r}}\,\cancel{r}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix} = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}$$

lokale Basis in Kugelkoordinaten:



$$\underline{\text{Abb}}: \quad \vec{r}: \quad \underbrace{\mathbb{R}_+ \times [0,\pi] \times [0,2\pi[}_{\subset\, \mathbb{R}^3} \longrightarrow \mathbb{R}^3$$

$$(r, \vartheta, \varphi) \longmapsto \vec{r} = r \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$$

$$\rightarrow \quad \vec{e}_r = \frac{1}{\left|\frac{\partial \vec{r}}{\partial r}\right|} \frac{\partial \vec{r}}{\partial r}$$

$$\vec{e}_\vartheta = \frac{1}{\left|\frac{\partial \vec{r}}{\partial \vartheta}\right|} \frac{\partial \vec{r}}{\partial \vartheta}$$

$$\vec{e}_\varphi = \frac{1}{\left|\frac{\partial \vec{r}}{\partial \varphi}\right|} \frac{\partial \vec{r}}{\partial \varphi}$$