Wdhlg.:

- Bahn: $t \longmapsto \vec{r}(t)$
- Geschwindigkeit $\vec{v}(t) = \dot{\vec{r}}(t)$
- Beschleunigung $\vec{a}(t) = \dot{\vec{v}}(t) = \ddot{\vec{r}}(t)$

Newton: $m\,\vec{a}(t) \overset{!}{=} \vec{F}(t)$

Falls $\vec{F} = \vec{F}(\vec{r})$ (Kraftfeld)

$\rightarrow$ Bewegungsgleichung

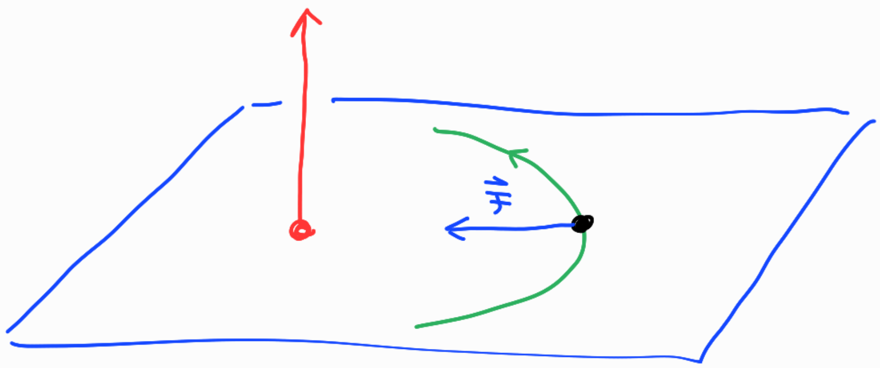
$$\boxed{m\,\ddot{\vec{r}}(t) = \vec{F}(\vec{r}(t))}$$

$\Rightarrow$ Bahnen $\vec{r}(t)$!

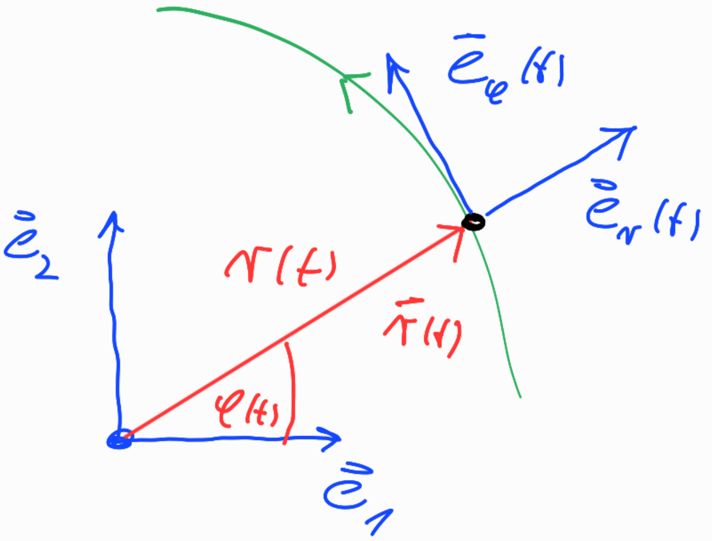
Ausblick: $\vec{F}(\vec{r})$ Zentralkraftfeld: $\vec{F}(\vec{r}) = f(r)\,\hat{r}$

$\rightarrow$ Drehimpulserhaltung

$\rightarrow$ Bahn $\vec{r}(t)$ verläuft in Ebene $\perp \vec{L}$



Bahn, Geschwindigkeit, Beschleunigung in Polarkoordinaten



zeitabhängige Zylinderkoordinaten

$$r(t)\,,\quad \varphi(t)$$

zeitabhängige lokale

ONB $\vec e_r(t),\ \vec e_\varphi(t)$

genauer:

$$\vec e_r(t) = \begin{pmatrix} \cos\varphi(t) \\ \sin\varphi(t) \end{pmatrix}, \quad \vec e_\varphi(t) = \begin{pmatrix} -\sin\varphi(t) \\ \cos\varphi(t) \end{pmatrix}$$

$$\vec r(t) = r(t)\,\vec e_r(t)$$

Geschw.:

$$\vec v(t) = \dot{\vec r}(t) = \frac{d}{dt}\left(r(t)\,\vec e_r(t)\right)$$

$$= \dot r(t)\,\vec e_r(t) + r(t)\,\dot{\vec e}_r(t)$$

kürzer:

$$\vec v = \dot r\,\vec e_r + r\,\dot{\vec e}_r$$

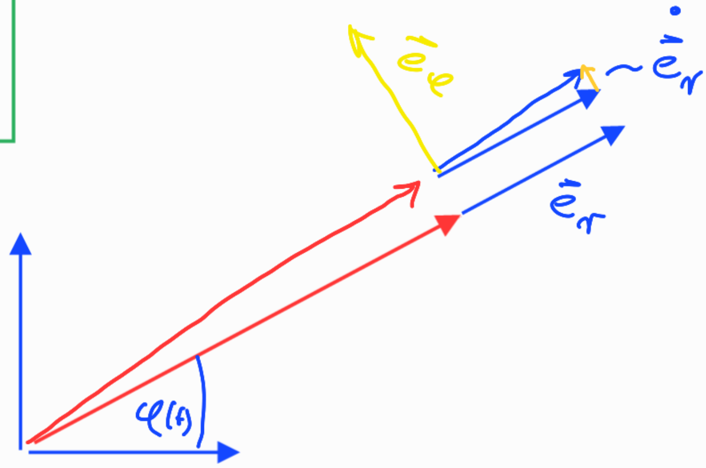
$$\longrightarrow \quad \vec a = \dot{\vec v} = \ddot r\,\vec e_r + 2\dot r\,\dot{\vec e}_r + r\,\ddot{\vec e}_r$$

$$\dot{\vec e}_r\,,\ \dot{\vec e}_\varphi\,,\ \ddot{\vec e}_r = ?$$

$$\vec e_r(t) = \begin{pmatrix}\cos\varphi(t)\\ \sin\varphi(t)\end{pmatrix}, \quad \vec e_\varphi(t) = \begin{pmatrix}-\sin\varphi(t)\\ \cos\varphi(t)\end{pmatrix}$$

$$\dot{\vec e}_r = \frac{d}{dt}\begin{pmatrix}\cos\varphi(t)\\ \sin\varphi(t)\end{pmatrix} = \begin{pmatrix}-\sin\varphi(t)\cdot\dot\varphi(t)\\ \cos\varphi(t)\cdot\dot\varphi(t)\end{pmatrix}$$

d.h. $$\boxed{\dot{\vec e}_r = \dot\varphi\,\vec e_\varphi}$$



analog:

$$\boxed{\dot{\vec e}_\varphi = -\dot\varphi\,\vec e_r}$$

$$\ddot{\vec e}_r = \frac{d}{dt}\left(\dot\varphi\,\vec e_\varphi\right) = \ddot\varphi\,\vec e_\varphi + \dot\varphi\,\underbrace{\dot{\vec e}_\varphi}_{=\,-\dot\varphi\,\vec e_r}$$

$$\boxed{\ddot{\vec e}_r = -\dot\varphi^2\,\vec e_r + \ddot\varphi\,\vec e_\varphi}$$

$$\vec v = \dot r\,\vec e_r + r\,\dot{\vec e}_r$$

$$\rightarrow \boxed{\vec v = \dot r\,\vec e_r + r\,\dot\varphi\,\vec e_\varphi}$$

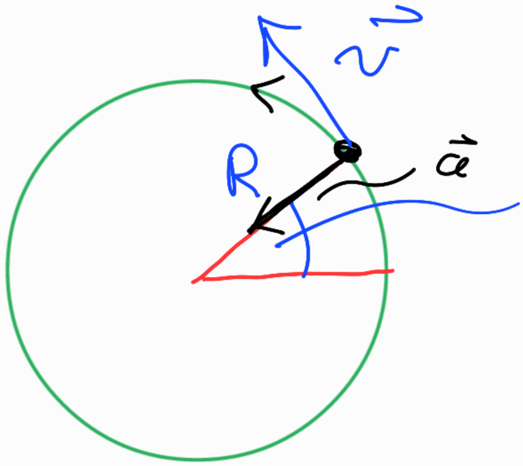
Radial- Azimuthalgeschw.

$$\vec a = \ddot r \, \vec e_r + 2 \dot r \, \dot{\vec e}_r + r \, \ddot{\vec e}_r$$

$$\dot{\vec e}_r = \dot\varphi \, \vec e_\varphi \qquad \ddot{\vec e}_r = -\dot\varphi^2 \, \vec e_r + \ddot\varphi \, \vec e_\varphi$$

$$\boxed{\vec a = \underbrace{(\ddot r - r\dot\varphi^2)}_{\text{Radial-}} \vec e_r + \underbrace{(2\dot r \dot\varphi + r \ddot\varphi)}_{\text{Azimuthalbeschl.}} \vec e_\varphi}$$

Bsp: gleichf. Kreisbew.



$r(t) = R$

$\varphi(t) = \omega t$

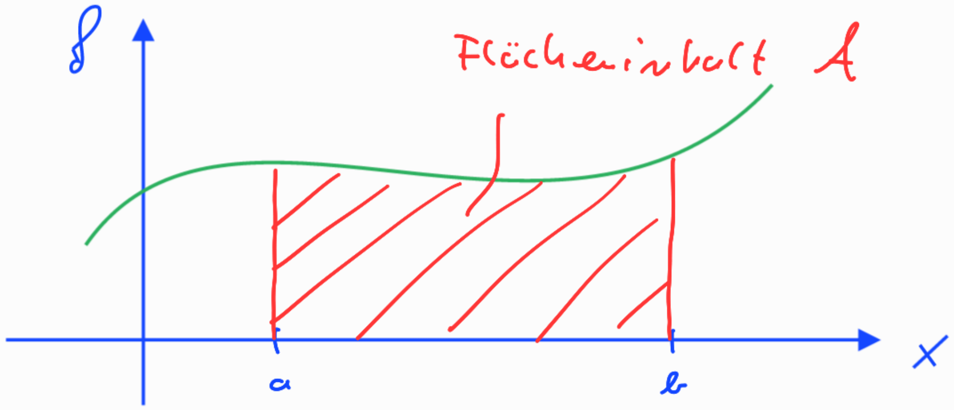
- $\vec v = R \omega \, \vec e_\varphi$
- $\vec a = -R\omega^2 \, \vec e_r$

Integral einer Fkt. f in den Grenzen a und b :

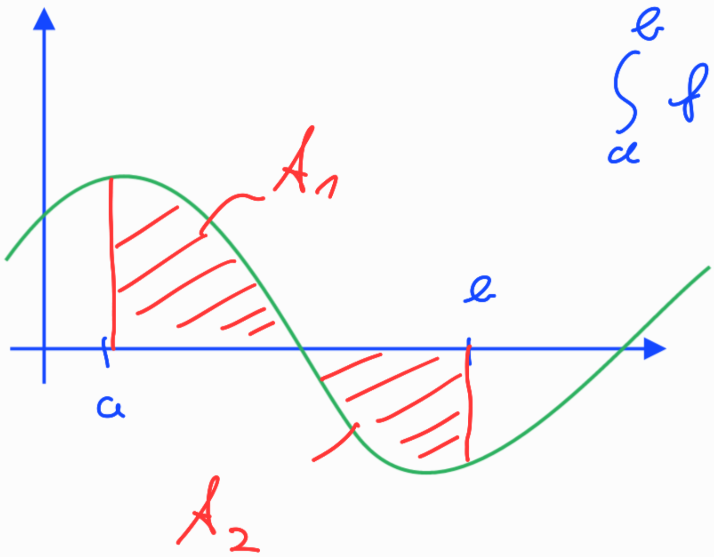
$$\int_a^b \underbrace{f(x)}_{\text{Integrand}} \, dx := A$$

Integrationsvariable

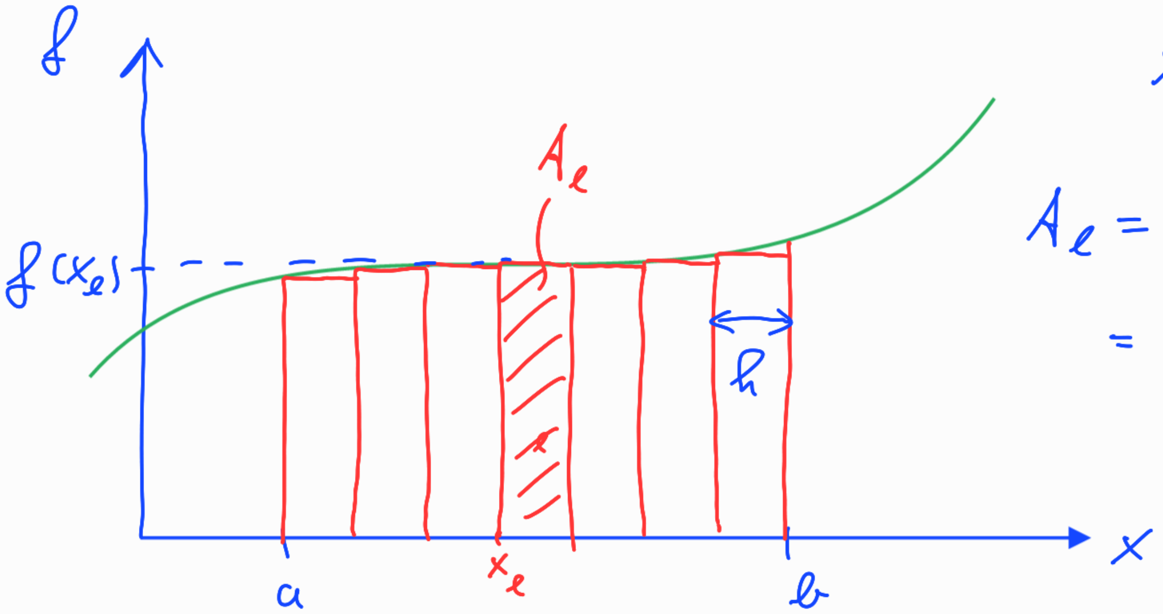
geometrisch:



$f \lessgtr 0$



$$\int_a^b f(x)\,dx := A_1 - A_2$$



$$x_l = a + hl$$

$$A_l = h\, f(x_l)$$
$$= h\, f(a + hl)$$

$$\boxed{\int_a^b f(x)\,dx := \lim_{h\to 0} \sum_l A_l = \lim_{h\to 0} \sum_{l=0}^{\frac{b-a}{h}} \underbrace{f(a+hl)}\, h}$$

$$\int_a^b f(x)\,dx$$

Eigenschaften:

1)
$$\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$$

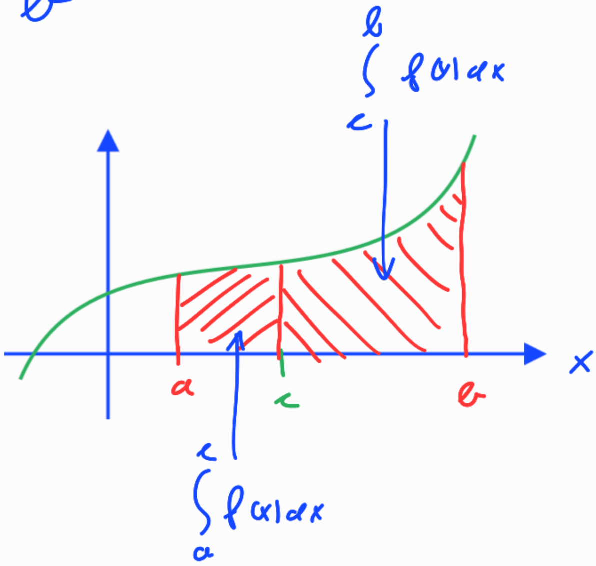
$$\int_a^b \left(f(x) + g(x)\right) dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(Linearität)

falls $b < a$:

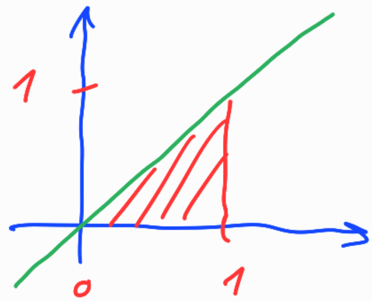
Def:
$$\int_a^b f(x)\,dx := -\int_b^a f(x)\,dx$$

2)
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

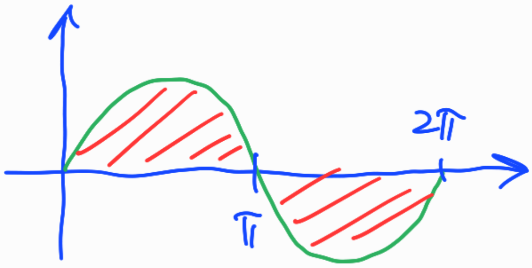


Bspe:

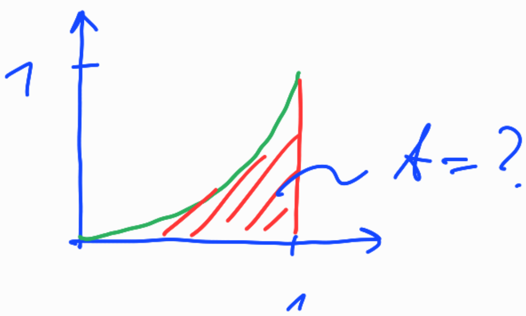
- $$\int_0^1 x\,dx = 1/2$$



- $$\int_0^{2\pi} \sin x\,dx = 0$$



- $$\int_0^1 x^2\,dx = 1/3$$



Hauptsatz der Differenzial- und Integralrechnung (HDI)

Def:

$$\boxed{\begin{array}{l} F \text{ Stammfunktion von } f \\ :\Longleftrightarrow \quad F' = f \end{array}}$$

(i) mit $F(x)$ auch $\tilde{F}(x) := F(x) + c$

Stammfunktion von f ✓

(ii) sind F und $\tilde{F}$ Stammfunktionen

von f, dann $F(x) - \tilde{F}(x) = c$

$\ulcorner$

$g(x) := F(x) - \tilde{F}(x)$

$\to \quad g'(x) = F'(x) - \tilde{F}'(x) = f(x) - f(x) = 0\ !$

$\to \quad g(x)$ ist eine konstante Fkt !

$g(x) = c \quad \to \quad F(x) - \tilde{F}(x) = c \quad \lrcorner$

HDI:

1) $F_u(x) := \int_u^x f(\tilde{x})\, d\tilde{x}$ ist Stammfunktion von f

2) $\int_a^b f(x)\, dx = F(b) - F(a) =: F(x)\Big|_a^b$

wobei $F(x)$ Stammfkt. von f

$\Rightarrow$ • $\int_0^1 x^2\, dx \overset{HDI}{=} \frac{x^3}{3}\Big|_0^1 = 1/3$

$\llcorner = \left(\frac{x^3}{3}\right)' \rightarrow \frac{x^3}{3}$ St.fk. von x^2

• $\int_0^\pi \sin x\, dx = -\cos x\Big|_0^\pi = +1+1 = 2$

$\llcorner (-\cos x)' \rightarrow -\cos x$ St.fkt. von $\sin x$

• $\int_a^b \sin(kx)\, dx = -\frac{\cos(kx)}{k}\Big|_a^b = \frac{\cos(ka)-\cos(kb)}{k}$

$\llcorner = \left(-\frac{\cos(kx)}{k}\right)' \rightarrow -\frac{\cos kx}{k}$ St.fkt. von $\sin kx$

$$F \underset{\text{integrieren}}{\overset{\text{ableiten}}{\rightleftarrows}} f$$

f	h'	x^α	$1/x$	e^x	$\sin x$	$\cos x$	...
F	h	$\frac{x^{\alpha+1}}{\alpha+1}$	$\ln x$	e^x	$-\cos x$	$\sin x$	...

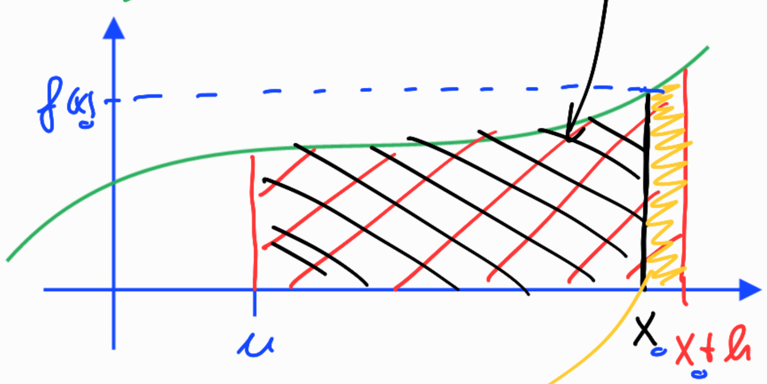
$\alpha \neq -1$

zum HDI: warum?

zu 1): $$F_u(x) = \int_u^x f(\tilde{x})\, d\tilde{x}$$

$$F_u'(x_0) = \frac{1}{h}\Big(F_u(x_0+h) - F_u(x_0) \Big) = f(x_0) \quad ✓$$

$h \to 0$



$\Delta F_u = h\, f(x_0)$

zu 2) : F Stammfunktion zu f

$$F(b) - F(a) = F_u(b) - F_u(a)$$

1) $F(x) = F_u(x) + c$

$$= \int_u^b f(x)\,dx - \int_u^a f(x)\,dx$$

$$= \int_a^u f(x)\,dx + \int_u^b f(x)\,dx = \int_a^b f(x)\,dx$$

Partielle Integration :

$$\boxed{\int_a^b f'(x)\,g(x)\,dx \overset{!}{=} f(x)\,g(x)\Big|_a^b - \int_a^b f(x)\,g'(x)\,dx}$$

$$= \int_a^b \{(f(x)\,g(x))' - f(x)\,g'(x)\}\,dx = f(x)\,g(x)\Big|_a^b - \int_a^b f(x)\,g'(x)\,dx$$

Substitutieregel

$$\boxed{\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(y))\cdot g'(y)\,dy \qquad x = g(y)}$$

$\ulcorner$ r.S. $= \displaystyle\int_{g^{-1}(a)}^{g^{-1}(b)} \underbrace{F'(g(y))\,g'(y)}_{=\frac{d}{dy}\left(F(g(y))\right)}\,dy \overset{\text{HDI}}{=} F(g(y))\Big|_{g^{-1}(a)}^{g^{-1}(b)}$.

F St. fkt. van f

$F(g(y))$ St. fkt. van $f(g(y))\,g'(y)$!

$$= F(b) - F(a) \overset{\text{HDI}}{=} \int_a^b f(x)\,dx \qquad \lrcorner$$