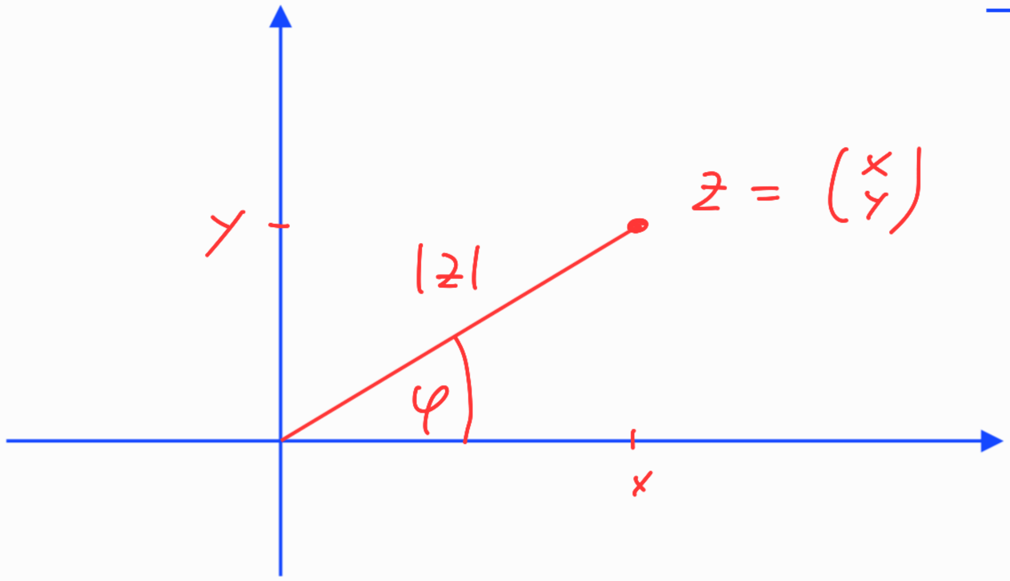
Komplexe Zahlen $\mathbb{C}$ = $\mathbb{R}^2$ mit Vektoraddition ✓ und Multiplikation !



$z = \begin{pmatrix} x \\ y \end{pmatrix}$

Realteil : $\operatorname{Re} z = x$

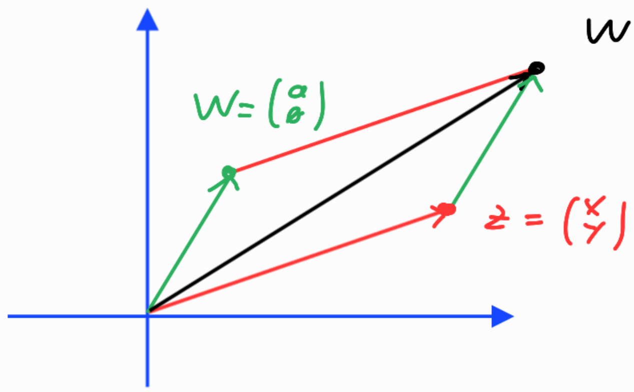
Imaginärteil : $\operatorname{Im} z = y$

Betrag (Modulus) : $|z| = \sqrt{x^2 + y^2}$

Argument : $\arg z = \varphi$

↳ Polardarstellung $\quad z = |z| \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$

Addition :



$w + z = \begin{pmatrix} \alpha + x \\ \beta + y \end{pmatrix}$

A1 – A4 ✓

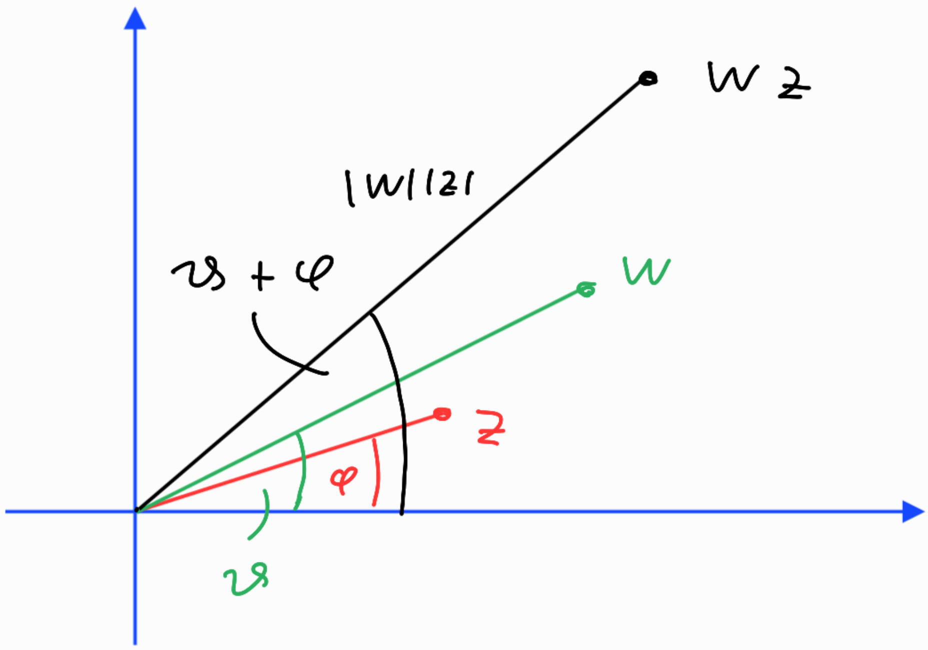
Multiplikation

$$w = \begin{pmatrix} a \\ b \end{pmatrix}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\leadsto \quad wz := \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix}$$

$$\rightarrow \quad |wz| = |w|\,|z|$$

$$\arg(wz) = \arg(w) + \arg(z)$$



- $wz = zw$

- $v(wz) = (vw)z$

- $\begin{pmatrix} 1 \\ 0 \end{pmatrix} z = z$

- z^{-1}: $\quad |z^{-1}| = |z|^{-1}$

  $\arg z^{-1} = -\arg z$

Distributivgesetz:

$$v(w+z) = vw + vz$$

$\rightarrow$ Rechne in $(\mathbb{C}, +, \cdot)$ genau wie in $(\mathbb{R}, +, \cdot)$!

- $\mathbb{R} \subset \mathbb{C}$ mittels $x \overset{!}{=} \begin{pmatrix} x \\ 0 \end{pmatrix}$

$\rightarrow$ Einselement $1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}$

Def.: imaginäre Einheit $\boxed{i := \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

$$\rightarrow \quad \underline{\underline{i^2}} = i \cdot i = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \underline{\underline{-1}}$$

$$\boxed{i^2 = -1}$$

$$\leadsto \boxed{i = \sqrt{-1}}$$

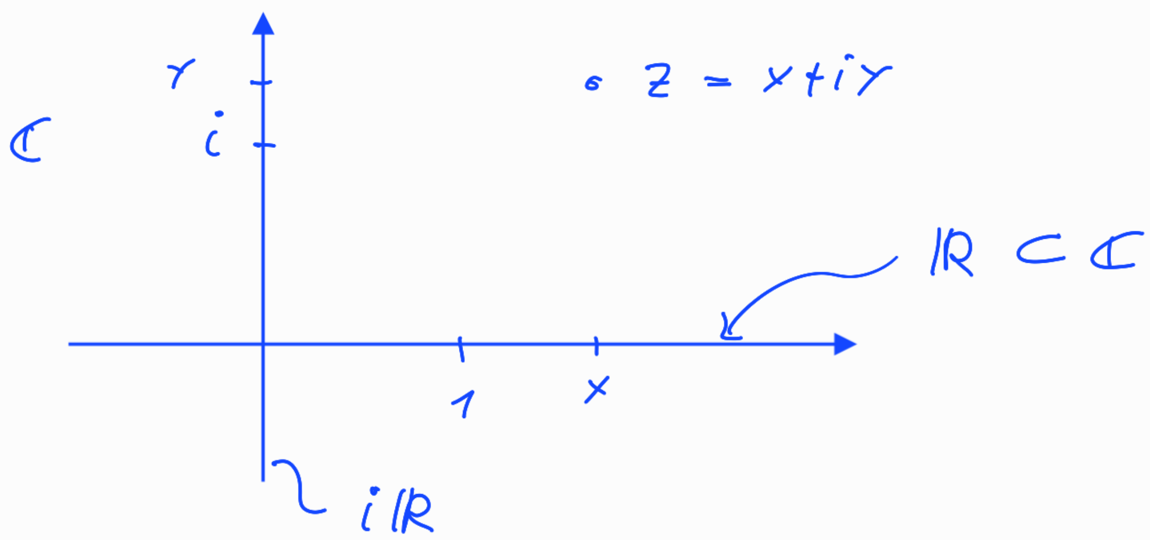
$$\rightarrow \quad z = \begin{pmatrix} x \\ y \end{pmatrix} = x \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{= 1} + y \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{= i} = x + iy$$

$$i^2 = -1 \quad \longrightarrow \quad w = a + ib, \quad z = x + iy$$

$$\longrightarrow \quad wz = (a+ib)(x+iy)$$

$$= \underline{ax} + \underline{iay} + \underline{ibx} + \overset{-1}{\overset{\shortparallel}{i^2}}\,\underline{by}$$

$$= ax - by + i(ay + bx)$$



Komplexe Lösungen algebraischer Gleichungen.

Bsp.:

$$z^2 + 2z + 3 = 0$$

$$\Longleftrightarrow \quad (z+1)^2 + 2 = 0$$

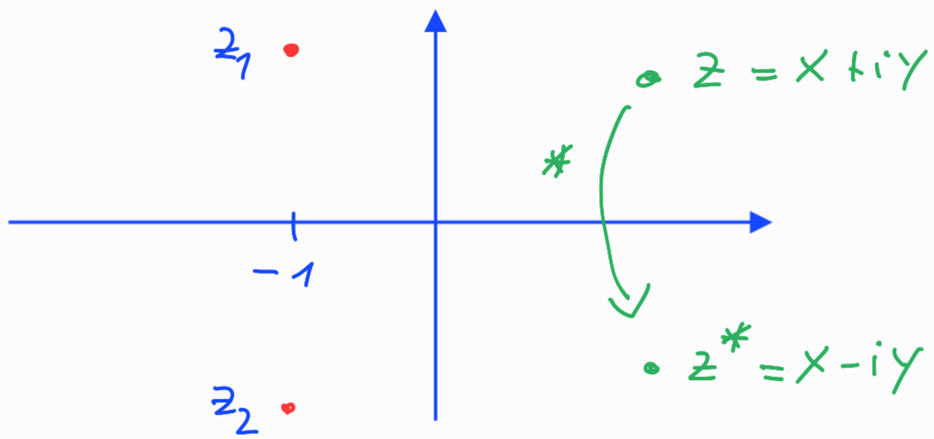
$$\Longleftrightarrow \quad (z+1)^2 = -2$$

$$\Longrightarrow \quad z_{1/2} + 1 = \pm\sqrt{-2} = \pm\sqrt{2}\,\overset{i}{\overset{\shortparallel}{\sqrt{-1}}}$$

$$\longrightarrow \quad z_{1/2} = -1 \pm i\sqrt{2}$$

Lsg.en der Gl. $z^2 + 2z + 3 = 0$:

$$z_{1/2} = \underline{-1} \pm \underline{i\sqrt{2}}$$



$$0 \stackrel{?}{=} z_1^2 + 2z_1 + 3 = (-1 + i\sqrt{2})^2 + 2(-1 + i\sqrt{2}) + 3$$

$$= \cancel{1} - \cancel{2i\sqrt{2}} \cancel{-2} \cancel{-2} + \cancel{2i\sqrt{2}} \cancel{+3} = 0!$$

<u>Komplexe Konjugation</u>

$\hat{=}$ Spiegelung an reeller Achse

$$z = x + iy \xrightarrow{*} z^* := x - iy$$

$$(= \bar{z})$$

$\Rightarrow$:

- $(z + w)^* = z^* + w^*$
- $(z w)^* = z^* w^*$
- $(z^*)^* = z$
- $\left(\frac{1}{z}\right)^* = \frac{1}{z^*}$

nützliche Beziehungen: $\left(z = x+iy \xrightarrow{*} z^* = x-iy \right)$

$\rightarrow$

- $\mathrm{Re}\, z = \frac{1}{2}(z + z^*)$
- $\mathrm{Im}\, z = \frac{1}{2i}(z - z^*)$
- $|z|^2 = z^* z$

$$\ulcorner \quad z^* z = (x-iy)(x+iy) = x^2 + y^2 = |z|^2 \quad \lrcorner$$

- $\frac{1}{z} = \frac{1}{z}\frac{z^*}{z^*} = \frac{z^*}{|z|^2}$

$$\ulcorner \quad (2+3i)^{-1} = \frac{1}{2+3i} = \frac{2-3i}{13} \quad \lrcorner$$

Komplexe Funktionen:

Frage: $e^z = e^{x+iy} = ?$,

$$e^z = \exp(z) = ?$$

$\ulcorner$ Erinnerung: $\exp(x) \overset{!}{=} \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$$\left(= \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\overbrace{(\exp(x))^{(n)}\Big|_{x=0}}^{=1}}_{e^x|_{x=0} = 1} \cdot x^n \right) \lrcorner$$

$\rightarrow$ Def:

$$\boxed{\exp(z) := \sum_{n=0}^{\infty} \frac{1}{n!} z^n}$$

alle reelle Fkten mit Reihendarstellung können auf diese Weise für kompl. Zahlen definiert werden:

$$\boxed{\begin{aligned} \sin(z) &:= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)!} z^{2\ell+1} \\ \cos(z) &:= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell)!} z^{2\ell} \end{aligned}}$$ ✓ ✓

Frage: wie berechnen wir $\exp(x+iy) = e^{x+iy}$?

$$e^{x+iy} = e^x \, e^{iy} \; ?$$

$$e^{iy} = \exp(iy) = \sum_{n=0}^{\infty} \frac{1}{n!} (iy)^n$$

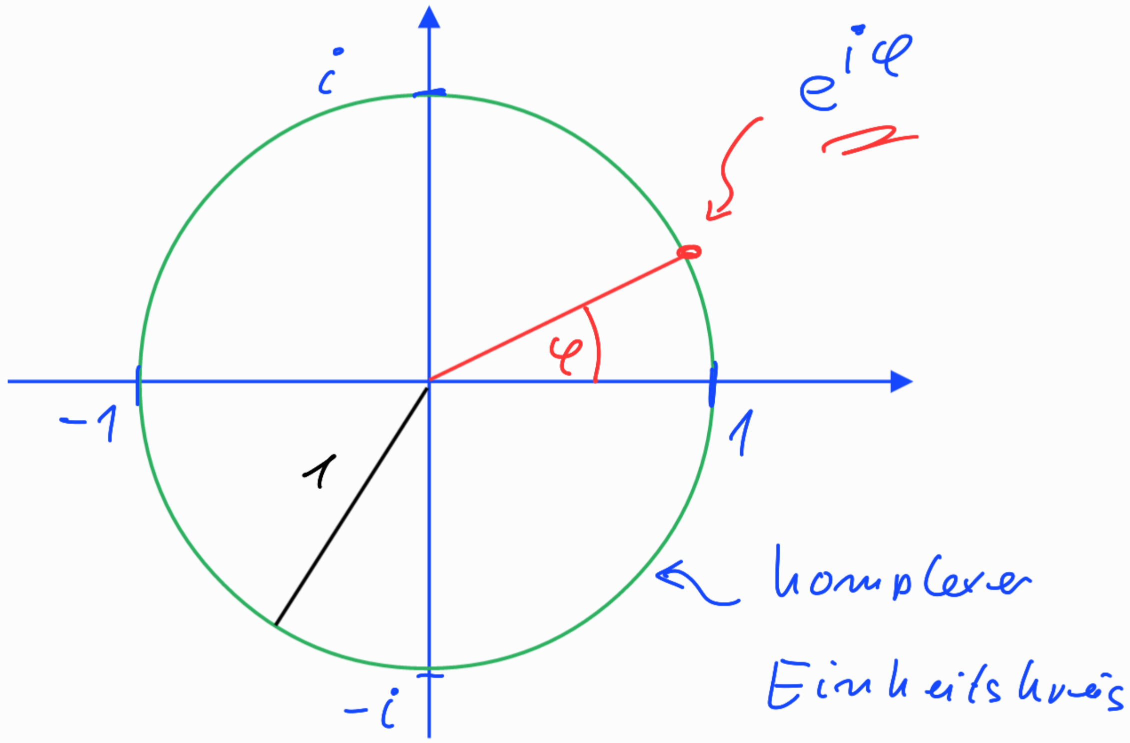
$$= \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!} \underbrace{(iy)^{2\ell}}_{= (i^2)^\ell y^{2\ell}} + \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)!} \underbrace{(iy)^{2\ell+1}}_{= i (i^2)^\ell y^{2\ell+1}}$$

$$= \underbrace{\sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell)!} y^{2\ell}}_{= \cos(y)} + \underbrace{i \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)!} y^{2\ell+1}}_{i \sin(y)} \quad !$$

d.h. $e^{iy} = \cos y + i \sin y$

<u>Euler-Identität</u>

$$\boxed{e^{i\varphi} = \cos\varphi + i\sin\varphi} = \begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix}$$



Anwendungen:

- Polardarstellung $\quad z = |z| \, e^{i\varphi}, \quad \varphi = \arg z$
- $i = e^{i\pi/2}, \quad -1 = e^{i\pi}, \quad -i = e^{i3\pi/2}$

$$1 = e^0 = e^{2\pi i} = e^{4\pi i} = \dots = e^{2\pi m i}$$

$$i^i \overset{!}{=} \left(e^{i\pi/2}\right)^i = e^{\underbrace{ii}_{=-1}\pi/2} = e^{-\pi/2}$$

$i = e^{i\pi/2}$

- Lösungen der Gl. $z^n \overset{!}{=} 1$

$\Rightarrow$ n-ten Einheitswurzeln:

Ansatz: $z = e^{i\varphi}$

$$\rightarrow \quad \underbrace{\left(e^{i\varphi}\right)^n}_{\textstyle e^{i\varphi n}} = 1$$

$$e^{i\varphi n} = 1 \overset{!}{=} e^{2\pi i \cdot m}, \qquad m \in \mathbb{Z}$$

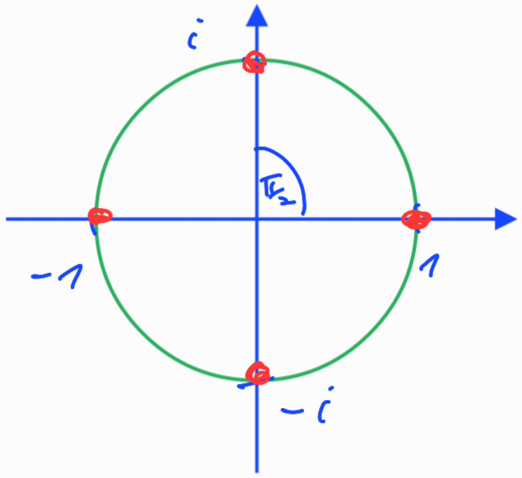
d.h. $\varphi n \overset{!}{=} 2\pi m$

$$\rightarrow \quad \boxed{\varphi_m = \frac{2\pi}{n} \cdot m \quad , \quad m = 0, 1, 2, \dots, n-1,}$$

Bsp.: $n=4$: $\quad z^4 = 1 \to \varphi_m = \frac{2\pi}{4}\cdot m$

$\to \quad \varphi_0 = 0, \quad \varphi_1 = \frac{\pi}{2}, \quad \varphi_2 = \pi, \quad \varphi_3 = \frac{3\pi}{2}$

$\to \quad e^{i\varphi_0} = 1, \quad e^{i\pi/2} = i, \quad e^{i\pi} = -1, \quad e^{i\frac{3\pi}{2}} = -i$



- <u>Additionstheoreme</u>: $\cos(\varphi+\vartheta) = \dots$?
  $\sin(\varphi+\vartheta) = \dots$?

!

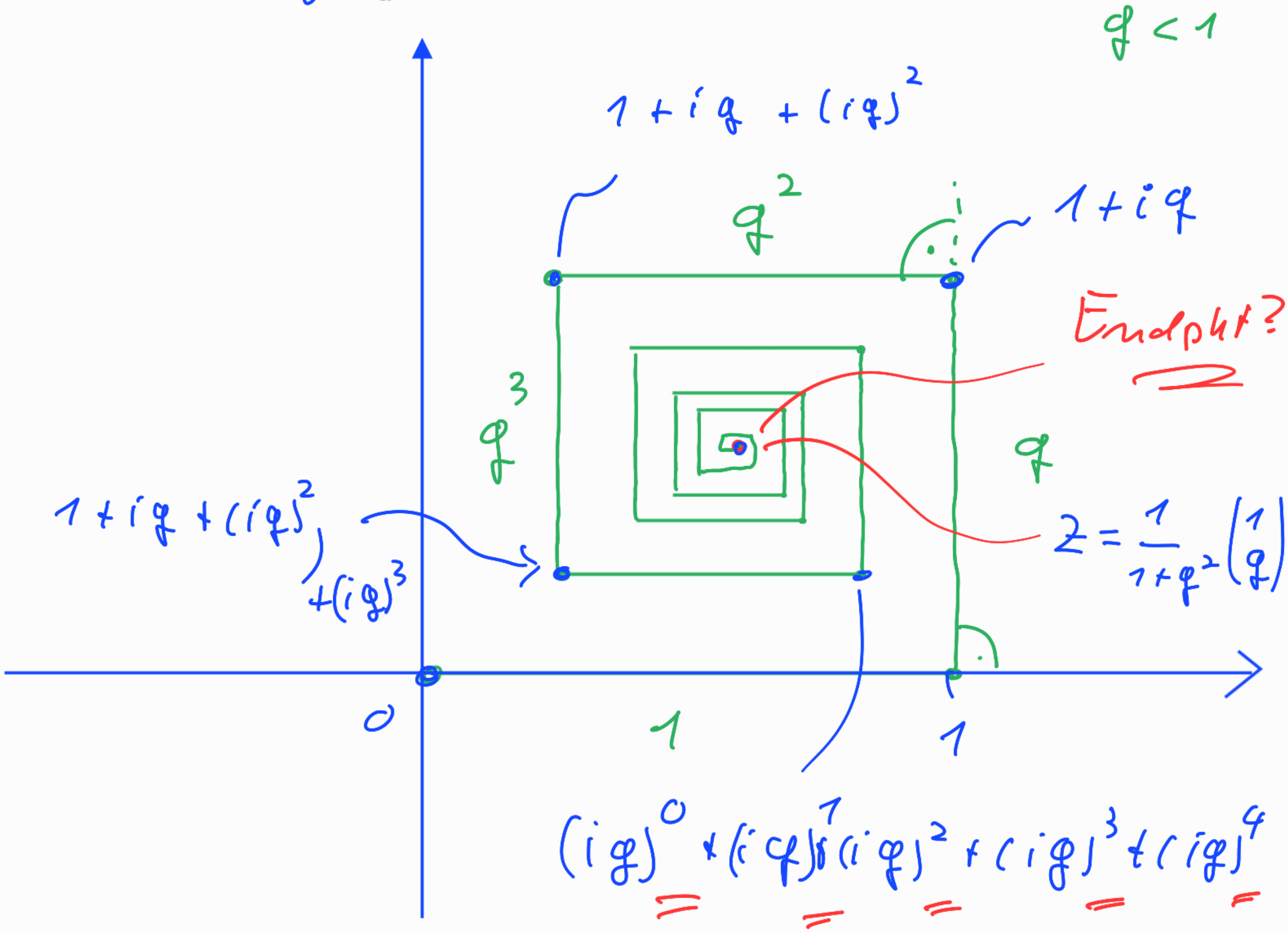
$$e^{i(\varphi+\vartheta)} \overset{!}{=} e^{i\varphi}\, e^{i\vartheta}$$

$$\boxed{\cos(\varphi+\vartheta)} + i\,\boxed{\sin(\varphi+\vartheta)} = (\cos\varphi + i\sin\varphi)\cdot(\cos\vartheta + i\sin\vartheta)$$

$$= \boxed{\cos\varphi\cos\vartheta - \sin\varphi\sin\vartheta}$$

$$+ i\,\boxed{(\cos\varphi\sin\vartheta + \sin\varphi\cos\vartheta)}$$

Problem: Weg eines "Euler"-Läufers in der Ebene:



$$z = \sum_{n=0}^{\infty} (iq)^n = \frac{1}{1-iq} = \frac{1+iq}{1+q^2}$$

geometrische Reihe

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$\left(\frac{1}{1-x} \overset{!}{=} \sum_{n=0}^{\infty} x^n \right)$$

Taylor um $x=0$