gestern: harmon. Oszill. mit externer Kraft:

$$\ddot{x} + \gamma \dot{x} + \omega^2 x = f_0 \cos \Omega t \qquad (*)$$

f_0: Stärke

Ω: ext. Frequenz

allg. Lsg.:

$$x(t) = x_h(t) + x_s(t)$$

$x_s(t)$: spez. Lsg. von $(*)$

$x_h(t)$: allg. Lsg. der homogenen DGL ✓ ($f_0 = 0$)

spezielle Lsg.:

$$x_s(t) = \mathrm{Re}\, a\, e^{i\Omega t} = |a| \cos(\Omega t + \alpha)$$

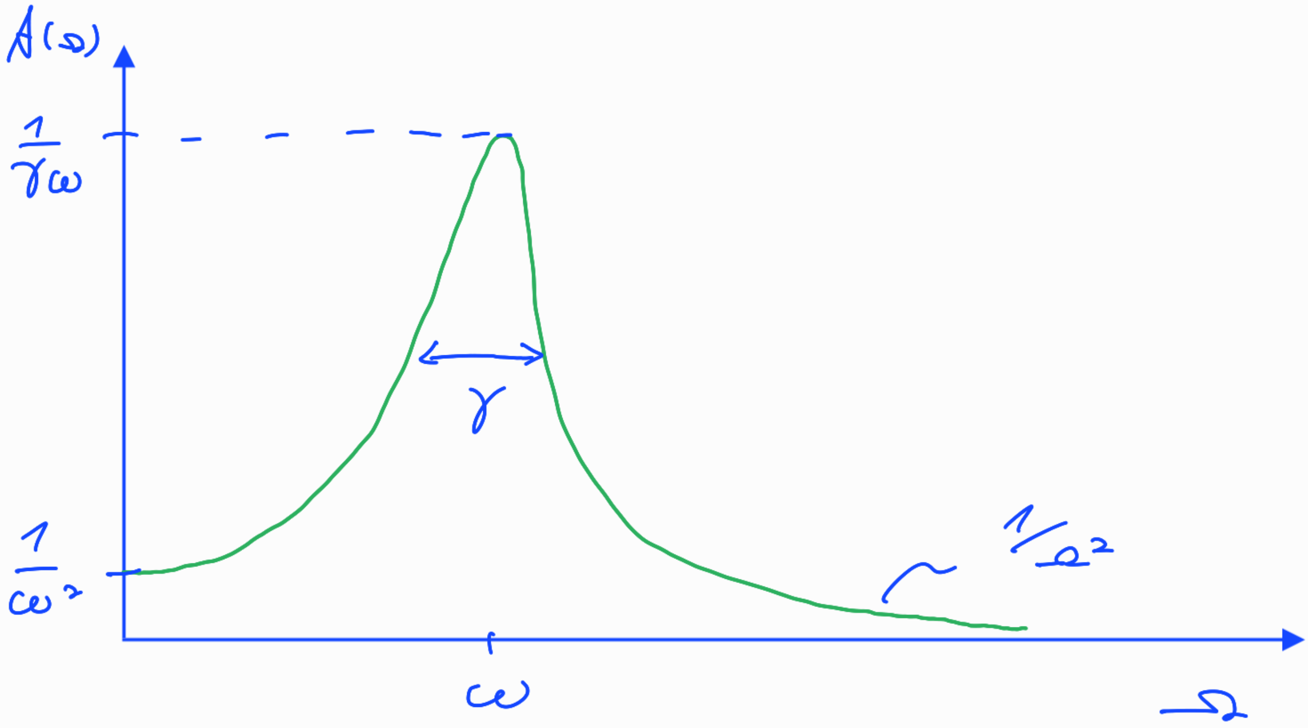
$$\text{mit} \qquad a = \frac{f_0}{\omega^2 - \Omega^2 + i\gamma\Omega}$$

Response-Funktion:

$$A_{\omega,\gamma}(\Omega) = |a| / f_0 \quad :$$

$$\rightarrow \quad A_{\omega,\gamma}(\Omega) = \left| \frac{1}{\omega^2 - \Omega^2 + i\gamma\Omega} \right|$$

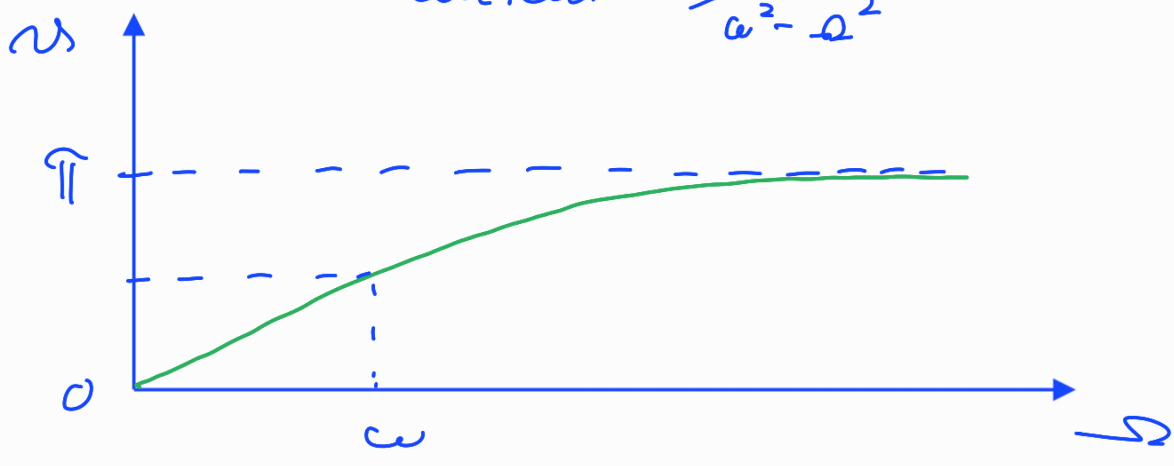
$$= \frac{1}{\sqrt{(\omega^2-\Omega^2)^2 + \gamma^2\Omega^2}}$$



Phasenverschiebung:

$$\vartheta = -\varphi = \arg(\omega^2 - \Omega^2 + i\gamma\Omega)$$

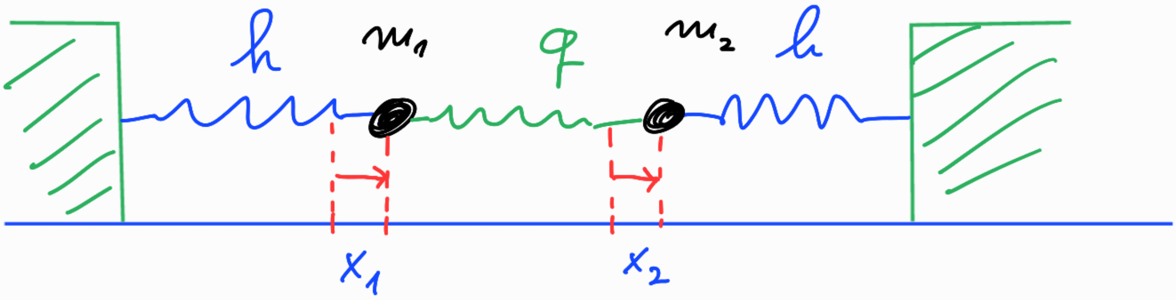
$$= \arctan \frac{\gamma\Omega}{\omega^2 - \Omega^2}$$



Eigenschwingungen und Eigenfrequenzen eines mech. Systems mit $n \geq 1$ Freiheitsgraden ($\rightarrow$ Eigenwerte, Eigenvektoren)

Bsp.

a) gekoppelte harm. Oszill.en:



$$F_1 = -k\,x_1 - q\,(x_1 - x_2)$$

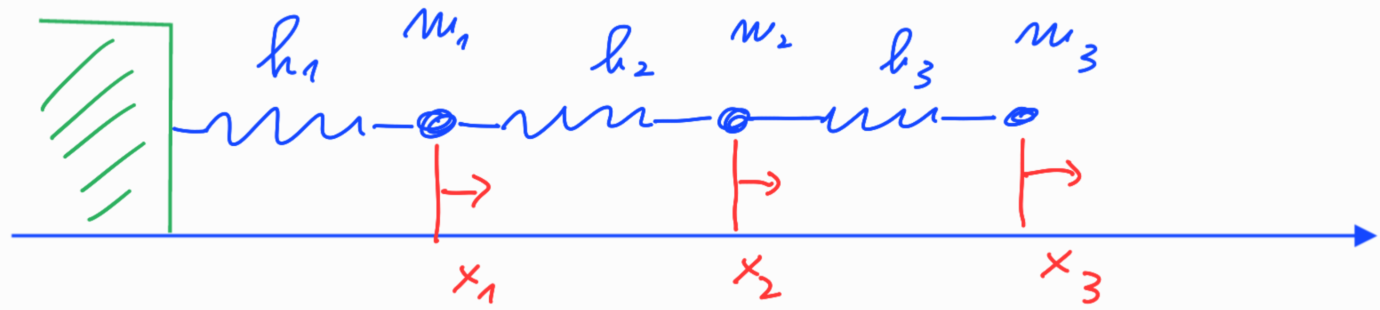
$$F_2 = -k\,x_2 - q\,(x_2 - x_1)$$

Newt. Bwgl.en: $\quad m_1 \ddot{x}_1 = F_1 \;, \quad m_2 \ddot{x}_2 = F_2$

$$\boxed{\begin{aligned} \ddot{x}_1 &= -\frac{k+q}{m_1}\,x_1 + \frac{q}{m_1}\,x_2 \\ \ddot{x}_2 &= \frac{q}{m_2}\,x_1 - \frac{k+q}{m_2}\,x_2 \end{aligned}}$$

lineares DGL-System $\rightarrow$ $x_1(t)$, $x_2(t)$

b)



$$F_1 = -k_1 x_1 - k_2 (x_1 - x_2)$$

$$F_2 = -k_2 (x_2 - x_1) - k_3 (x_2 - x_3)$$

$$F_3 = -k_3 (x_3 - x_2)$$

N. G. en:

$$\boxed{\begin{aligned} \ddot{x}_1 &= -\frac{k_1 + k_2}{m_1} x_1 + k_2 x_2 \\ \ddot{x}_2 &= \frac{k_2}{m_2} x_1 - \frac{k_2 + k_3}{m_2} x_2 + \frac{k_3}{m_2} x_3 \\ \ddot{x}_3 &= + k_3 x_2 - k_3 x_3 \end{aligned}}$$

lin. DGL-System $\rightarrow$ $x_1(t)$

$x_2(t)$

$x_3(t)$

analytisch, Bsp a:

$m_1 = m_2 = m$

DGL:

$$\boxed{\begin{aligned} \ddot{x}_1 &= -\frac{k+q}{m} x_1 + \frac{q}{m} x_2 \\ \ddot{x}_2 &= \frac{q}{m} x_1 - \frac{k+q}{m} x_2 \end{aligned}}$$

→ Auslenkung: $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

$\frac{d}{dt} \downarrow$

Geschw. $\dot{\vec{x}}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix}$

$\frac{d}{dt} \downarrow$

Beschl. $\ddot{\vec{x}}(t) = \begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix}$

$$\ddot{\vec{x}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} \frac{k+q}{m} x_1 & -\frac{q}{m} x_2 \\ -\frac{q}{m} x_1 & + \frac{k+q}{m} x_2 \end{pmatrix}$$

$$= -\underbrace{\begin{pmatrix} \frac{k+q}{m} & -\frac{q}{m} \\ -\frac{q}{m} & \frac{k+q}{m} \end{pmatrix}}_{K} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\vec{x}}$$

$\uparrow$

„Hooksche Matrix“

lin. DGL-System:

$$(*) \quad \boxed{\ddot{\vec{x}}(t) = -K \cdot \vec{x}(t)}$$

ähnlich

$$\ddot{x}(t) = -\frac{k}{m} x(t)$$

$$\downarrow$$

$$x(t) = a\, e^{i\omega t}$$

bestimme Lsg. $\vec{x}(t)$ mittels

<u>Exponentialansatz</u>

$$\boxed{\vec{x}(t) = \vec{u}\; e^{i\omega t}}$$

Frequenz!

Vektor (≙ Auslenkung)

in DGL (*) einsetzen:

- $$\ddot{\vec{x}} = \frac{d^2}{dt^2}\left(\vec{u}\, e^{i\omega t}\right) = -\omega^2 \vec{u}\, e^{i\omega t}$$

$$\ddot{\vec{x}} = -\omega^2 \vec{u}\, e^{i\omega t}$$

$$\boxed{\cancel{-}\,\omega^2 \vec{u}\, \cancel{e^{i\omega t}} \overset{!}{=} \cancel{-}\,K \vec{u}\, \cancel{e^{i\omega t}}}$$

DGL

$\vec{x}(t) = \vec{u}\, e^{i\omega t}$ Lsg. der DGL $\ddot{\vec{x}} = -K\vec{x}$

g. d. w.

$$\boxed{K \underline{\vec{u}} = \underline{\lambda} \vec{u}}$$

$$\lambda = \omega^2$$

Eigenwertproblem!

Def:

$m \times n$ Matrix $A = \underbrace{\left.\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{pmatrix}\right\} m}_{n}$

m: Anzahl Zeilen

n: Anzahl Spalten

kurz $A = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$

Matrix-Vektor-Multiplikation:

$\mathbb{R}^m \ni \underbrace{A}_{m \times n} \underbrace{\vec{x}}_{\mathbb{R}^n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$$:= \begin{pmatrix} a_{11} x_1 + a_{12} x_2 + \cdots a_{1n} x_n \\ a_{21} x_1 + a_{22} x_2 + \cdots a_{2n} x_n \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots a_{mn} x_n \end{pmatrix}$$

kurz:

$$(A\vec{x})_l := \sum_{i=1}^{n} a_{li} x_i$$

$l: 1, \dots, m$

$\ulcorner$

- $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
- $A(\lambda\vec{x}) = \lambda A\vec{x}$

$$A = (a_{ij}) \quad , \quad B = (b_{ij})$$

$$\rightarrow \quad A + B := (a_{ij} + b_{ij})$$

$$\lambda A := (\lambda a_{ij})$$

- $(A+B)\vec{x} = A\vec{x} + B\vec{x}$ $\lrcorner$ $\lrcorner$

zurück zur DGL: $\ddot{\vec{x}} = -K\vec{x}$

$(m_1 = m_2 = m)$

$$K = \frac{1}{m}\begin{pmatrix} h+q & -q \\ -q & h+q \end{pmatrix}$$

$\vec{x}(t) = \vec{u}\, e^{i\omega t}$ Lsg. $\Longleftrightarrow$ $\boxed{K\vec{u} \overset{!}{=} \omega^2 \vec{u}}$

physik. Einsicht: $\vec{u}_I = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad \vec{u}_I\,!$

$$\rightarrow K\vec{u}_I = \frac{1}{m}\begin{pmatrix} h+q & -q \\ -q & h+q \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{m}\begin{pmatrix} h \\ h \end{pmatrix} = \frac{h}{m}\underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\vec{u}_I}$$

d.h. $\boxed{K\vec{u}_I = \frac{h}{m}\vec{u}_I}$ $\rightsquigarrow$ $\omega_I = \sqrt{\frac{h}{m}}$