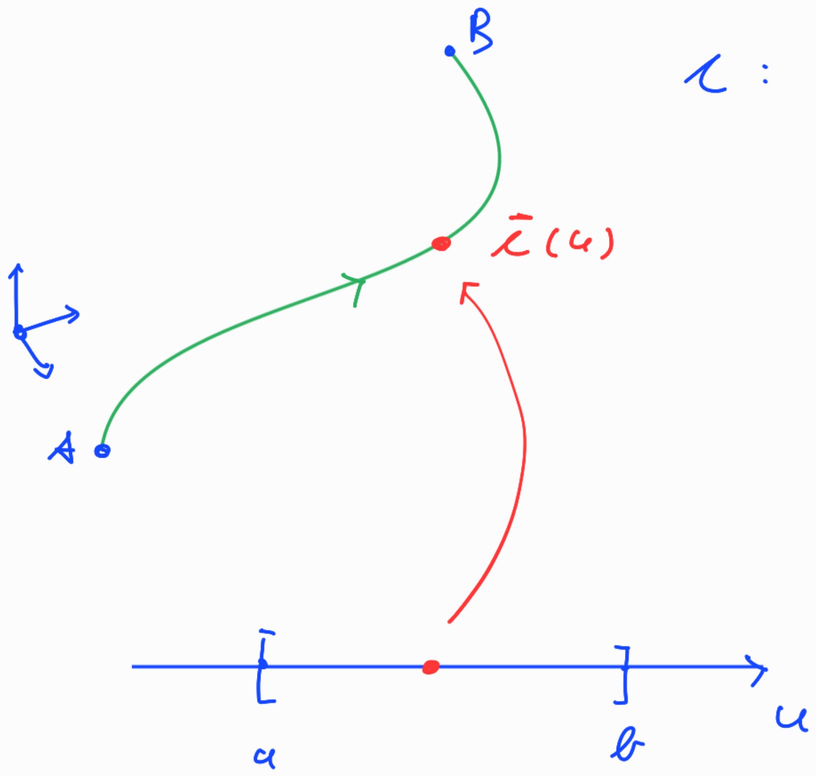
gestern:

Kurve c : Parametrisierung = diff. Abb.

$$c : [a, b] \longrightarrow \mathbb{R}^n$$
$$u \longmapsto \vec{c}(u)$$



- Länge:

$$\boxed{L(c) = \int_c dl := \int_a^b |\vec{c}\,'(u)|\, du}$$

- Integral des Vektorfelds $\vec{A} : \vec{r} \mapsto \vec{A}(\vec{r})$

längs c :

$$\boxed{\int_c \vec{A} \cdot d\vec{l} := \int_a^b \langle \vec{A}(\vec{c}(u)), \vec{c}\,'(u) \rangle\, du}$$

$\ulcorner$ $\vec{F}(\vec{r}) = -\mathrm{grad}\,(V(\vec{r}))$

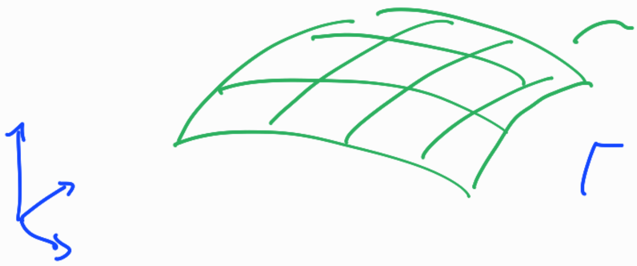
$$\frac{d}{du}\Big(V(\vec{c}(u))\Big) = \langle \mathrm{grad}\, V(\vec{c}(u)), \vec{c}\,'(u)\rangle$$

$$\int_c \mathrm{grad}\, V \cdot d\vec{l} \overset{\text{Def}}{=} \int_a^b \underbrace{\langle \mathrm{grad}\, V(\vec{c}(u)), \vec{c}\,'(u)\rangle}_{\|}$$

$$= \int_a^b \frac{d}{du} V(\vec{c}(u))\, du$$

$$\underset{\text{HDI}}{\overset{!}{=}} V(\vec{c}(u))\Big|_a^b \overset{!}{=} V(\vec{c}(b)) - V(\vec{c}(a)) \lrcorner$$

Flächenintegral



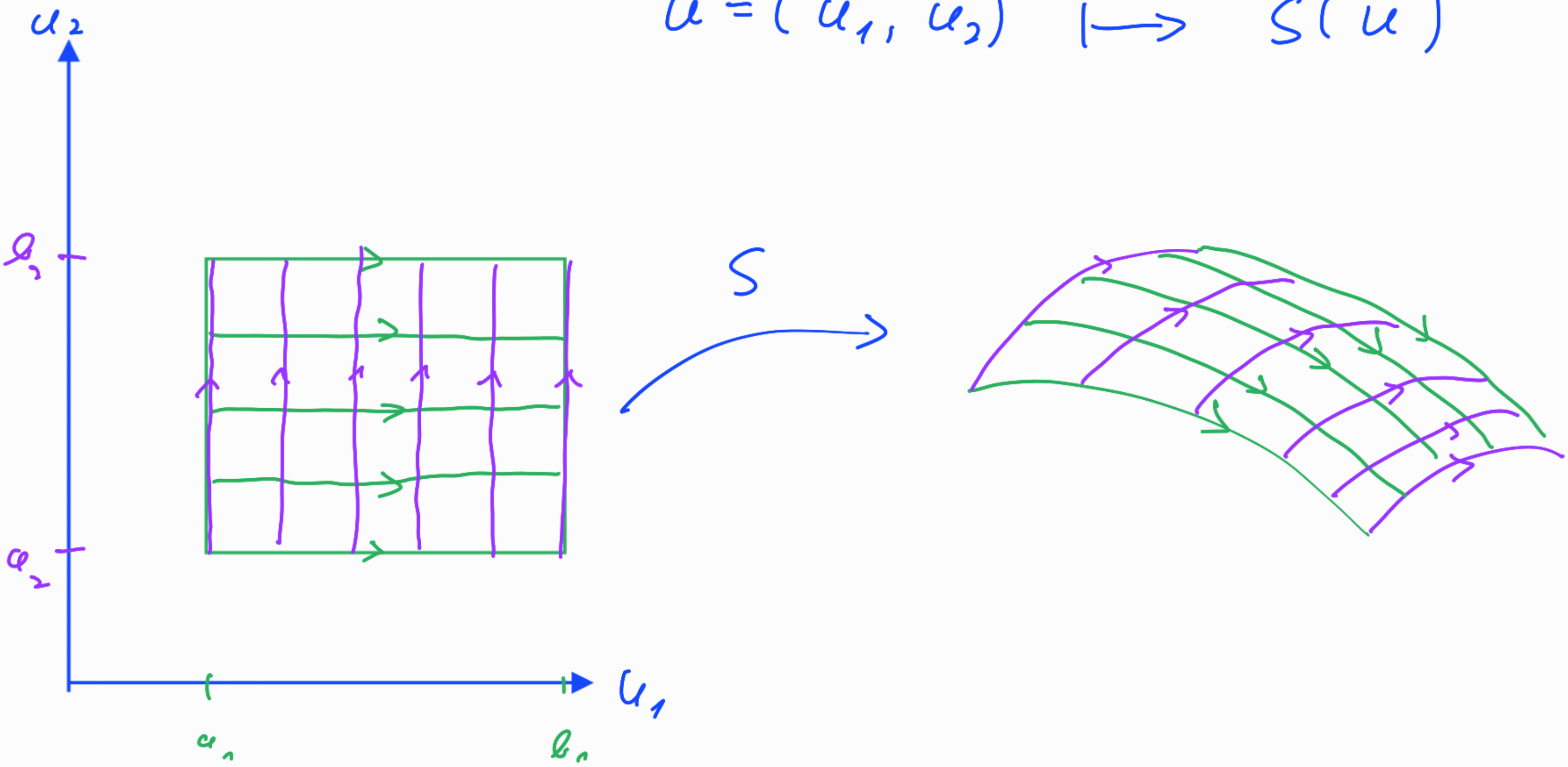
Fläche S

- Flächeninhalt von S?
- Strom I_S durch Fläche S?

Parametrisierung der Fläche S =

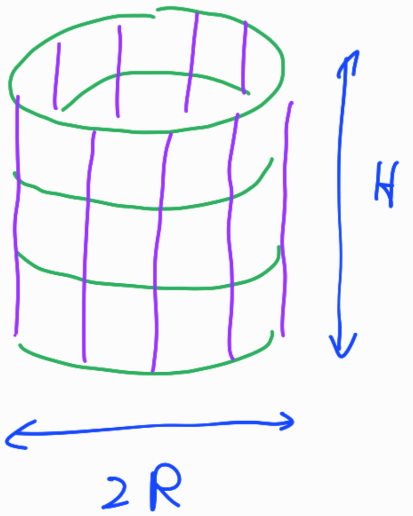
diff. Abb. $S : [a_1, b_1] \times [a_2, b_2] \longrightarrow \mathbb{R}^3$

$$\vec{u} = (u_1, u_2) \longmapsto \vec{S}(\vec{u})$$

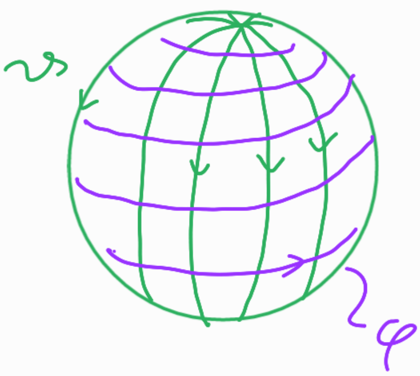


Bsp: Zylindermantel Z:

$$Z : [0, 2\pi] \times [0, H] \longrightarrow \mathbb{R}^3$$
$$(\varphi, z) \longmapsto \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix}$$



- Sphäre vom Radius R:



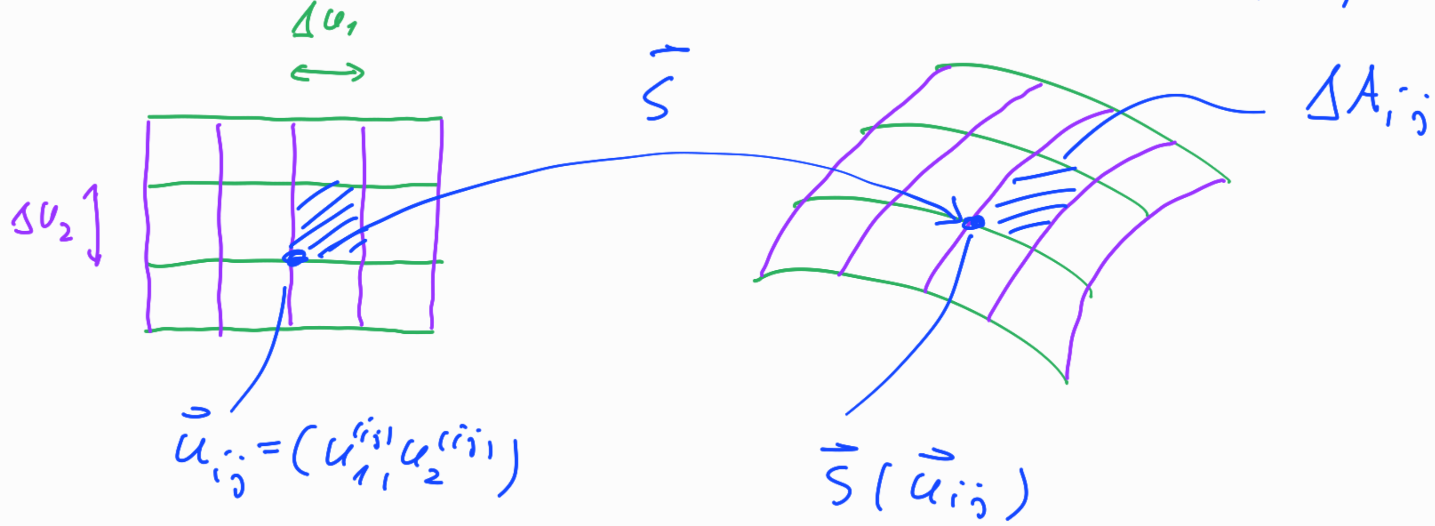
$$S : [0, \pi] \times [0, 2\pi] \longrightarrow \mathbb{R}^3$$
$$(\vartheta, \varphi) \longmapsto R \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix}$$

Flächeninhalt einer Fläche S

↳ Paramet.

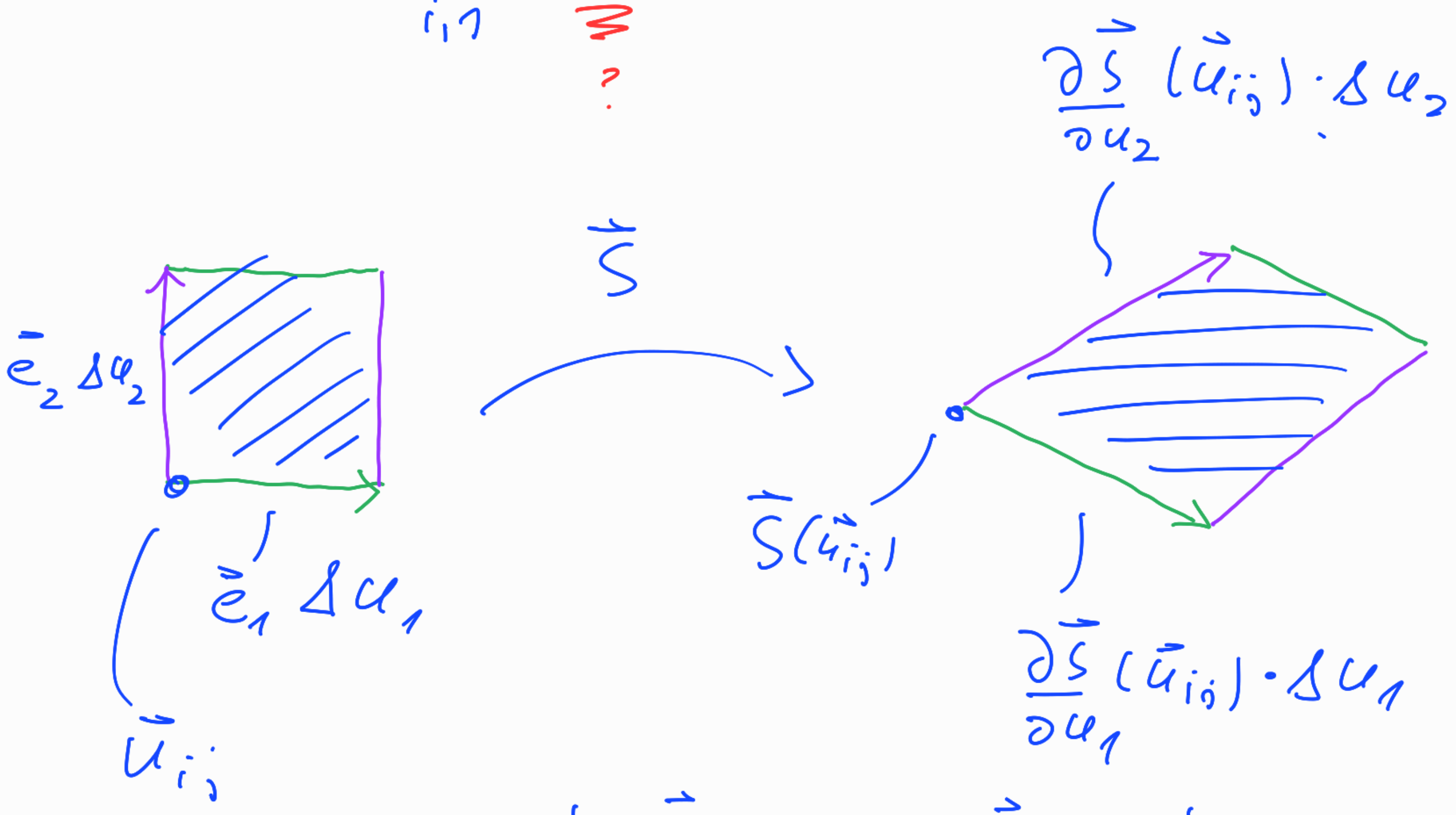
$\vec S : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}^3$

$\vec u = (u_1, u_2) \mapsto \vec S(\vec u)$



$$A(S) = \sum_{i,j} \Delta A_{ij}$$

$\approx$?



$$\longrightarrow \quad \Delta A_{ij} = \left| \frac{\partial \vec S}{\partial u_1}(\vec u_{ij}) \times \frac{\partial \vec S}{\partial u_2}(\vec u_{ij}) \right| \Delta u_1 \Delta u_2$$

$$\Delta A_{ij} = \left| \frac{\partial \vec{S}}{\partial u_1}(\vec{u}_{ij}) \times \frac{\partial \vec{S}}{\partial u_2}(\vec{u}_{ij}) \right| \Delta u_1 \Delta u_2$$

$$\leadsto A(S) = \sum_{i,j} \Delta A_{ij}$$

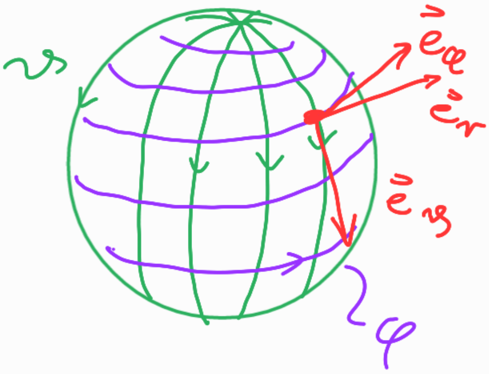
$$= \sum_{ij} \left| \frac{\partial \vec{S}}{\partial u_1}(\vec{u}_{ij}) \times \frac{\partial \vec{S}}{\partial u_2}(\vec{u}_{ij}) \right| \Delta u_1 \Delta u_2$$

$$\boxed{A(S) = \int_S |d\vec{f}\,| := \int_{a_1}^{b_1} \int_{a_2}^{b_2} \left| \frac{\partial \vec{S}}{\partial u_1}(\vec{u}) \times \frac{\partial \vec{S}}{\partial u_2}(\vec{u}) \right| du_1\, du_2}$$

$$\ulcorner\ A(S) = \int_S df = \iint_S df \quad \dots \ \lrcorner$$

Beispiel: Flächeninhalt der Sphäre vom Radius R:

Sphäre vom Radius R:



$$S : [0,\pi] \times [0,2\pi] \to \mathbb{R}^3$$

$$(\vartheta, \varphi) \mapsto R \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix}$$

$$\frac{\partial \vec{S}}{\partial \vartheta} \overset{!}{=} R \vec{e}_\vartheta \; ; \quad \frac{\partial \vec{S}}{\partial \varphi} = R \sin\vartheta \underbrace{\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}}_{\vec{e}_\varphi}$$

$$\frac{\partial \vec{S}}{\partial \varphi} = R \sin\vartheta \, \vec{e}_\varphi$$

$$\left| \frac{\partial \vec{S}}{\partial \vartheta} \times \frac{\partial \vec{S}}{\partial \varphi} \right| = R^2 \sin\vartheta \, |\underbrace{\vec{e}_\vartheta \times \vec{e}_\varphi}_{= \vec{e}_r \, !}| = R^2 \sin\vartheta$$

$$A(S) = \int_0^\pi \underbrace{\int_0^{2\pi} R^2 \sin\vartheta \, d\varphi}_{2\pi R^2 \sin\vartheta} \, d\vartheta$$

$$= \int_0^\pi 2\pi R^2 \sin\vartheta \, d\vartheta$$

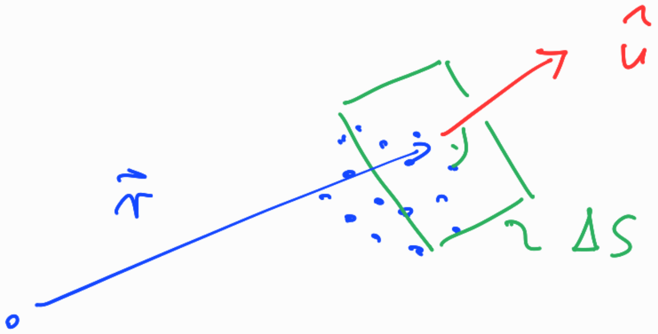
$$= 2\pi R^2 \underbrace{\int_0^\pi \sin\vartheta \, d\vartheta}_{(-\cos\vartheta)\big|_0^\pi = -\overbrace{\cos\pi}^{-1} + \overbrace{\cos 0}^{1} = 2} = 4\pi R^2$$

Flächenintegral eines Vektorfelds $\vec{A}$

über Fläche S:

$\Rightarrow$
- Teilchenstrom $I_N(S)$
- Ladungsstrom $I_q(S)$
- Volumenstrom $I(S)$

Stromdichte:



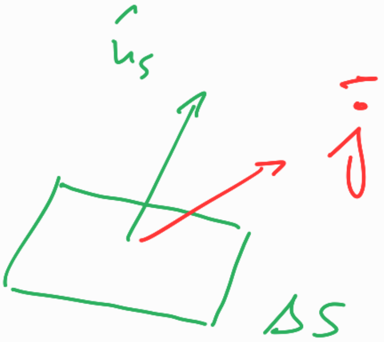
Teilchenstrom ΔI durch Fläche ΔS

(Teilchen/Zeit)

Stromdichte $$\boxed{\vec{j}(\vec{r}) := \frac{\Delta I}{\Delta S} \cdot \hat{n}}$$

$\Rightarrow$ allg. Flächenstück: ΔS

$$\left.\begin{array}{ll} \bullet \text{ Flächeninhalt} & \Delta S \\ \bullet \text{ Flächennormale} & \hat{n}_S \end{array}\right\} \rightarrow \Delta\vec{S} = \Delta S\, \hat{n}_S$$



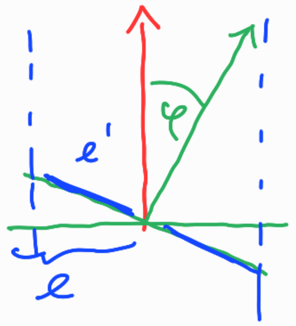
$$\boxed{I(\vec{\Delta S}) = \langle \vec{j}, \vec{\Delta S} \rangle}$$

(*)

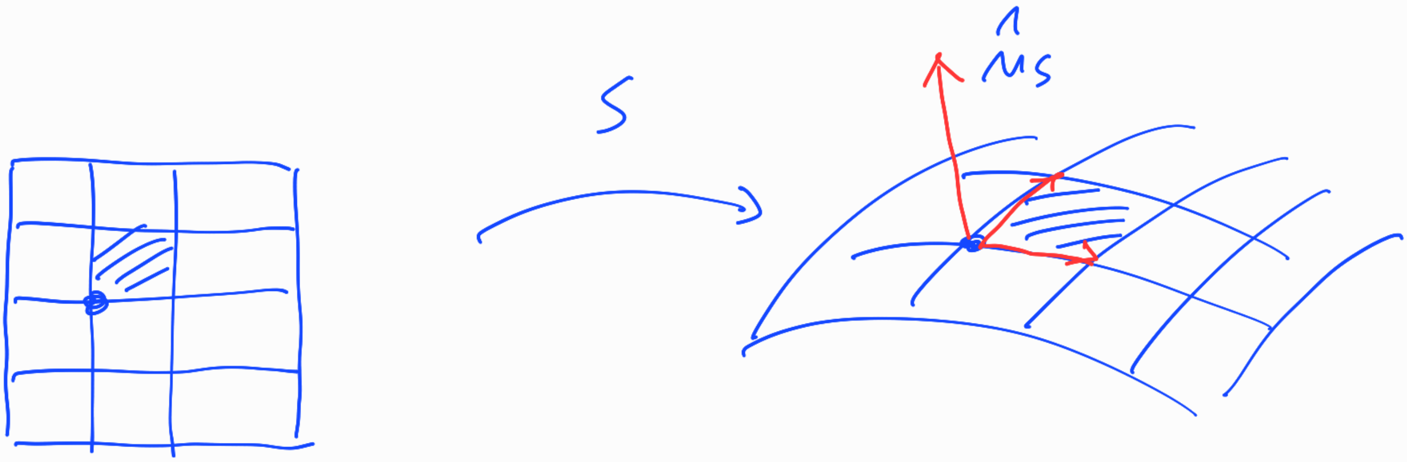
$$\rightarrow \quad I(\vec{\Delta S}) = \Delta S \, |\vec{j}| \cos\varphi$$

$$= \Delta S \, |\vec{j}| \langle \hat{j}, \hat{n}_S \rangle$$

$$= \langle \vec{j}, \vec{\Delta S} \rangle$$



$$\ell' = \ell \cos\varphi = \ell \langle \hat{j}, \hat{n}_S \rangle$$



$$\Delta S = \left| \frac{\partial \vec{S}}{\partial u_1} \times \frac{\partial \vec{S}}{\partial u_2} \right| \Delta u_1 \Delta u_2$$

$$\hat{n}_S = \widehat{\frac{\partial \vec{S}}{\partial u_1} \times \frac{\partial \vec{S}}{\partial u_2}}$$

$$\rightsquigarrow \quad \Delta\vec{S}_{ij} = \frac{\partial \vec{S}}{\partial u_1} \times \frac{\partial \vec{S}}{\partial u_2} \, \Delta u_1 \, \Delta u_2 \Big|_{\vec{u}_{ij}}$$

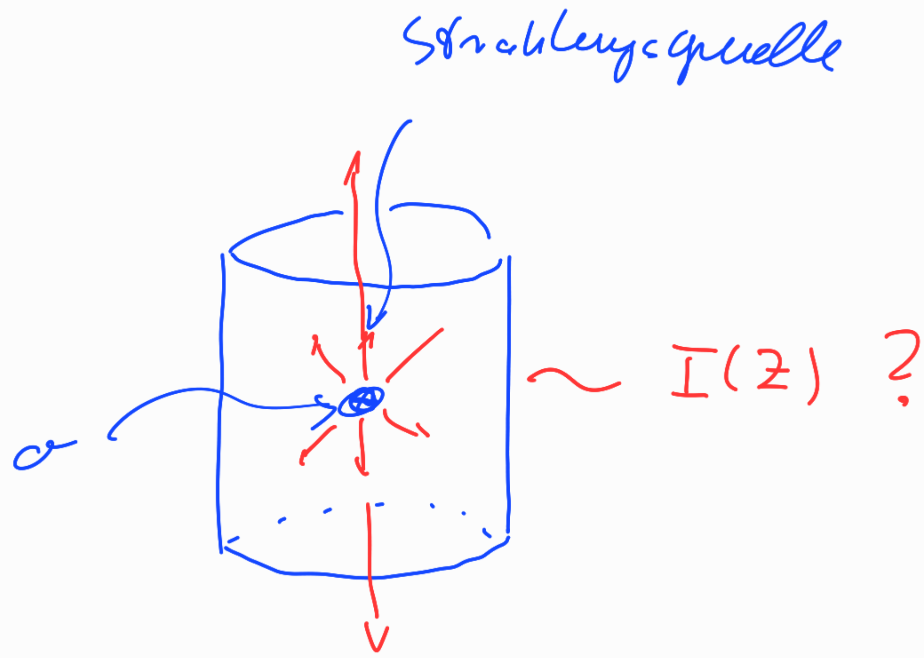
$$I(S) = \sum_{ij} I(\Delta\vec{S}_{ij})$$

$$\overset{(*)}{=} \sum_{ij} \left\langle \vec{\dot{j}}\left(\vec{S}(\vec{u}_{ij})\right), \frac{\partial \vec{S}}{\partial u_1}(\vec{u}_{ij}) \times \frac{\partial \vec{S}}{\partial u_2}(u_{ij}) \right\rangle \Delta u_1 \Delta u_2$$

$$I(S) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \left\langle \vec{\dot{j}}\left(\vec{S}(\vec{u})\right), \frac{\partial \vec{S}}{\partial u_1}(\vec{u}) \times \frac{\partial \vec{S}}{\partial u_2}(\vec{u}) \right\rangle du_1 \, du_2$$

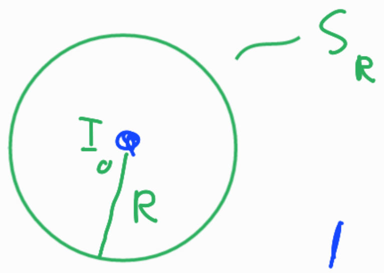
$$\boxed{\int_S \vec{A} \, d\vec{f} := \int_{a_1}^{b_1} \int_{a_2}^{b_2} \left\langle \vec{A}\left(\vec{S}(\vec{u})\right), \frac{\partial \vec{S}}{\partial u_1}(\vec{u}) \times \frac{\partial \vec{S}}{\partial u_2}(\vec{u}) \right\rangle du_1 \, du_2}$$

Beispiel:



Teilchenstromdichte der Punktquelle:

$$\boxed{\vec{j}(\vec{r}) = \frac{I_0}{4\pi}\,\frac{\hat{r}}{r^2}}$$



$$I(S_R) \overset{!}{=} I_0$$

$$I(S_R) = \int_0^{\pi}\int_0^{2\pi} f(R)\,\Big\langle \hat{r}, \underbrace{\frac{\partial \vec{S}}{\partial \vartheta} \times \frac{\partial \vec{S}}{\partial \varphi}}_{R^2 \sin\vartheta\, \vec{e}_r = R^2\sin\vartheta\,\hat{r}} \Big\rangle \, d\vartheta\, d\varphi$$

$$\int_{S_R} f(R)\, \underbrace{\hat{r}\, d\vec{f}}_{|d\vec{f}|} = f(R) \underbrace{\int_0^{\pi}\int_0^{2\pi} R^2 \sin\vartheta\, d\vartheta\, d\varphi}_{4\pi R^2}$$

$$= f(R)\, 4\pi R^2$$

Stromdichte

$$\vec{j}(\vec{r}) = \frac{I_0}{4\pi} \frac{\hat{r}}{r^2}$$