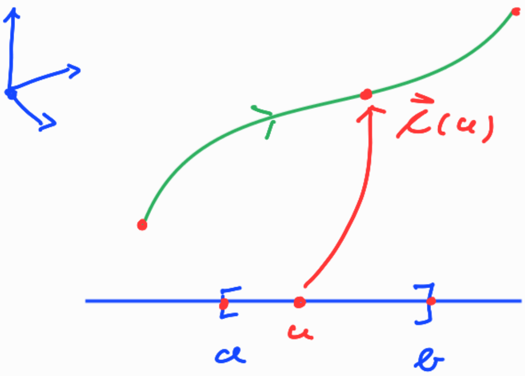
Wiederholung:

Kurve:

Parametrisierung: $c : [a, b] \to \mathbb{R}^3$

$\qquad\qquad\qquad u \mapsto \vec{c}(u)$



• <u>Länge</u> $\quad L(c) = \int_c dl := \int_a^b |\vec{c}\,'(u)|\, du$

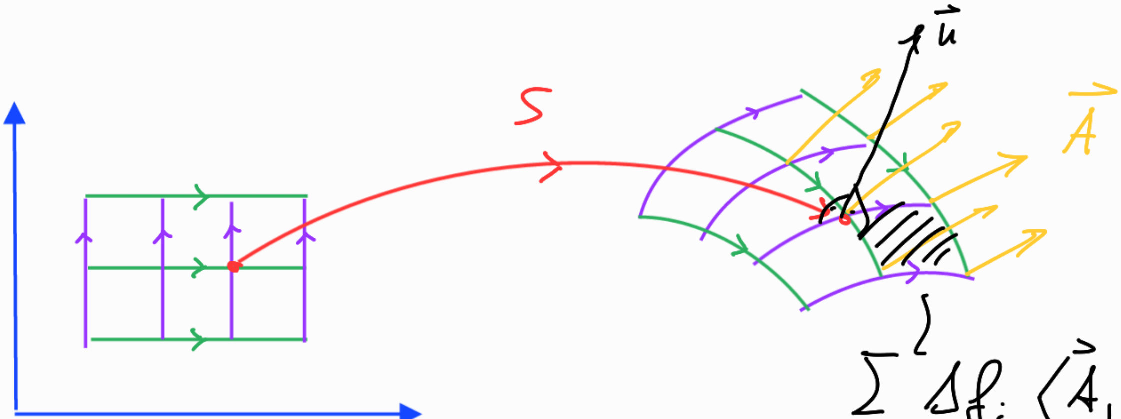
• <u>Integral von VF $\vec{A}$ längs c:</u>

$$\int_c \vec{A} \cdot \vec{dl} := \int_a^b \langle \vec{A}(\vec{c}(u)), \vec{c}\,'(u) \rangle \, du$$

Fläche:

Parametrisierung: $S : [a_1, b_1] \times [a_2, b_2] \longrightarrow \mathbb{R}^3$

$$\vec{u} = (u_1, u_2) \longmapsto \vec{S}(\vec{u})$$



$$\sum_i \Delta f_i \langle \vec{A}, \hat{n} \rangle = \int_S \vec{A}\, d\vec{f}$$

- ## Flächeninhalt:

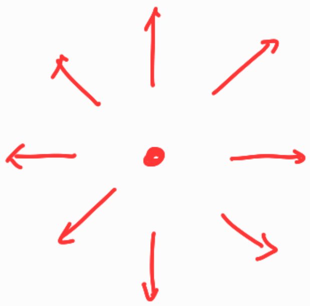
$$A(S) = \int_S df := \int_{a_1}^{b_1} du_1 \int_{a_2}^{b_2} du_2 \left| \frac{\partial \vec{S}}{\partial u_1} \times \frac{\partial \vec{S}}{\partial u_2} \right|$$

- ## Integral des VFs $\vec{A}$ über Fläche S:

$$\int_S \vec{A}\, d\vec{f} := \int_{a_1}^{b_1} du_1 \int_{a_2}^{b_2} du_2 \left\langle \vec{A}(\vec{S}(u)), \frac{\partial \vec{S}}{\partial u_1} \times \frac{\partial \vec{S}}{\partial u_2} \right\rangle$$

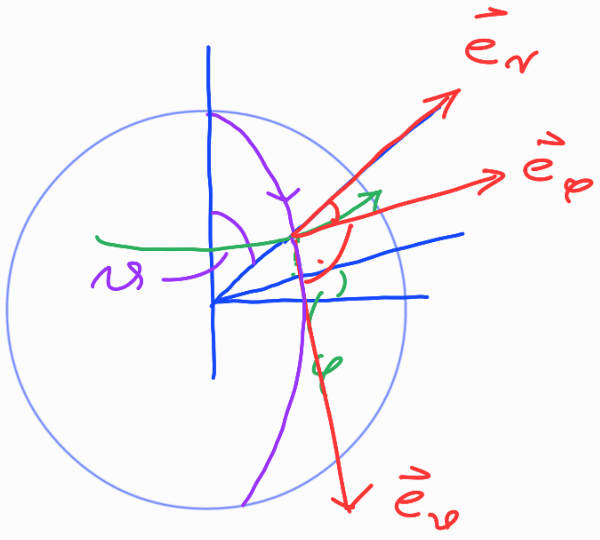
Beispiel

- Stromdichte einer punktförmigen Teilchenquelle in O, Gesamtstrom I_0:



$$\boxed{\vec{j}(\vec{r}) = \frac{I_0}{4\pi}\,\frac{\vec{e}_r}{r^2}}$$

- Strom durch Kugelschale, Radius R, Mittelpkt. O:



$$\frac{\partial \vec{S}}{\partial \vartheta} = \underline{\underline{R}}\,\vec{e}_\vartheta$$

$$\frac{\partial \vec{S}}{\partial \varphi} = \underline{\underline{R\sin\vartheta}}\,\vec{e}_\varphi$$

Parametrisierung:

$$S : [0,\pi] \times [0,2\pi]$$

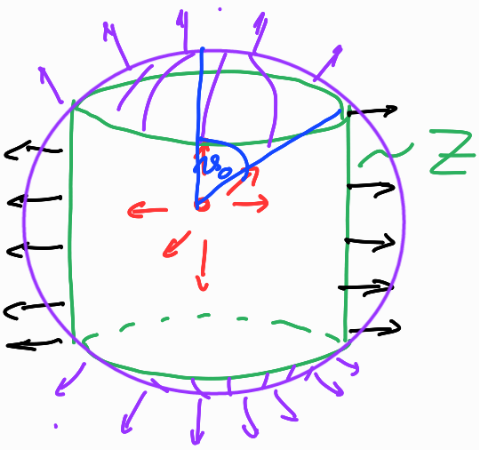
$$(\vartheta, \varphi) \longmapsto \vec{S}(\vartheta,\varphi) = R \underbrace{\begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}}_{\vec{e}_r}$$

$$\rightarrow \quad \frac{\partial \vec{S}}{\partial \vartheta} \times \frac{\partial \vec{S}}{\partial \varphi} = R^2 \sin\vartheta \; \vec{e}_r$$

$$\rightarrow \quad I(S) = \int_S \vec{j}\, d\vec{f} = \int_0^{\pi} \int_0^{2\pi} \underbrace{\left\langle \vec{j}, \frac{\partial \vec{S}}{\partial \vartheta} \times \frac{\partial \vec{S}}{\partial \varphi} \right\rangle}_{= \frac{I_0}{4\pi R^2} \cdot R^2 \sin\vartheta} d\vartheta\, d\varphi \qquad \vec{j}(\vec{r}) = \frac{I_0}{4\pi r^2}\, \vec{e}_r$$

$$= \frac{I_0}{4\pi} \int_0^{2\pi} \underbrace{\int_0^{\pi} \sin\vartheta\, d\vartheta}_{=2}\, d\varphi = I_0 \quad ✓$$

- Strom durch Zylindermantel:



$$I(Z) = I_0 - 2\, I_K$$

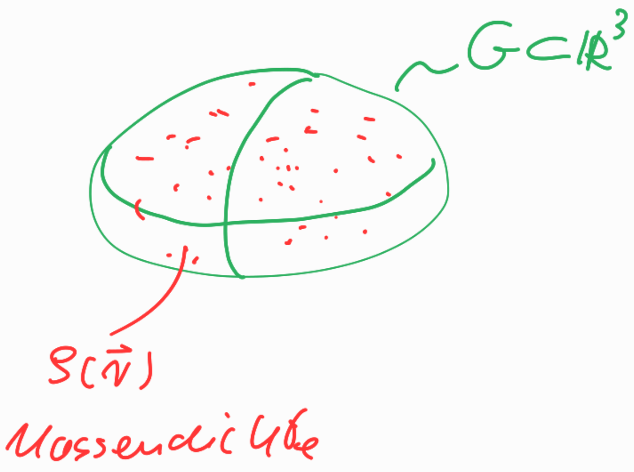
$$I_K = \int_0^{\vartheta_0} \int_0^{2\pi} \frac{I_0}{4\pi R_2} R^2 \sin\vartheta\, d\varphi\, d\vartheta$$

$$= \frac{I_0}{2} \int_0^{\vartheta_0} \sin\vartheta\, d\vartheta = \frac{I_0}{2}(1 - \cos\vartheta_0)$$

$$\rightarrow \quad I(Z) = I_0 \cos\vartheta_0$$

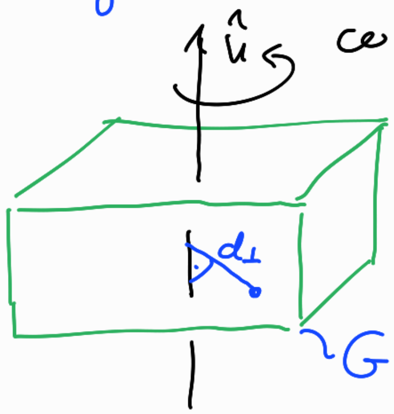
Volumengebiete und Volumenintegrale

Motivation: a)



$$M(G) = \int_G \rho \, dV$$

b) Trägheitsmoment $I_{\hat{u}}$:



Rotationsenergie:

$$E_{Rot} = \frac{1}{2} I_{\hat{u}} \, \omega^2$$

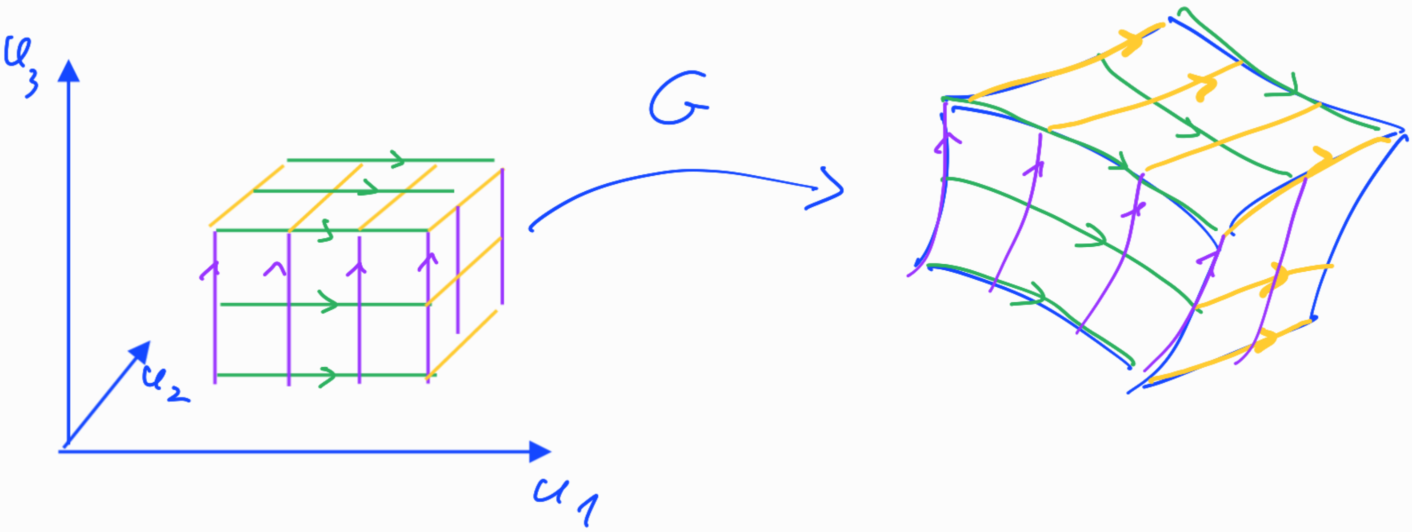
$$L_{\hat{u}} = I_{\hat{u}} \, \omega$$

$$I_{\hat{u}} = \int_G \rho \, d_\perp^2 \, dV \quad !$$

Parametrisierung eines Volumengebiets $G \subset \mathbb{R}^3$:

$$G : [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \longrightarrow \mathbb{R}^3$$

$$\vec{u} = (u_1, u_2, u_3) \longmapsto \vec{G}(\vec{u})$$

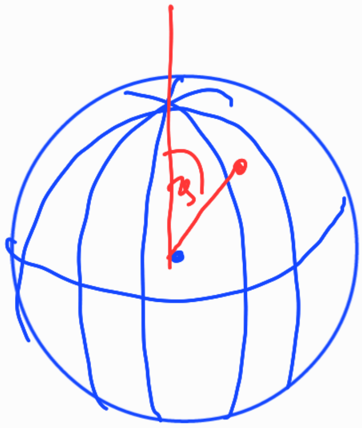


<u>Beispiele:</u> Vollzylinder Z

$$Z : [0, 2\pi] \times [0, R] \times [0, H] \longrightarrow \mathbb{R}^3$$

$$(\varphi, \rho, z) \longmapsto \vec{Z}(\varphi, \rho, z) = \begin{pmatrix} \rho \cos\varphi \\ \rho \sin\varphi \\ z \end{pmatrix}$$

Vollkugel



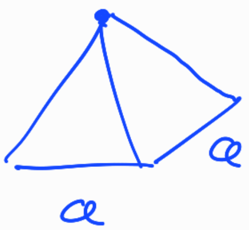
$$K : [0,\pi] \times [0,2\pi] \times [0,R] \mapsto \mathbb{R}^2$$

$$(\vartheta\ ,\ \varphi\ ,\ r)$$

$$\mapsto \vec{r}(\vartheta,\varphi,r) = r \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$$

Pyramide:

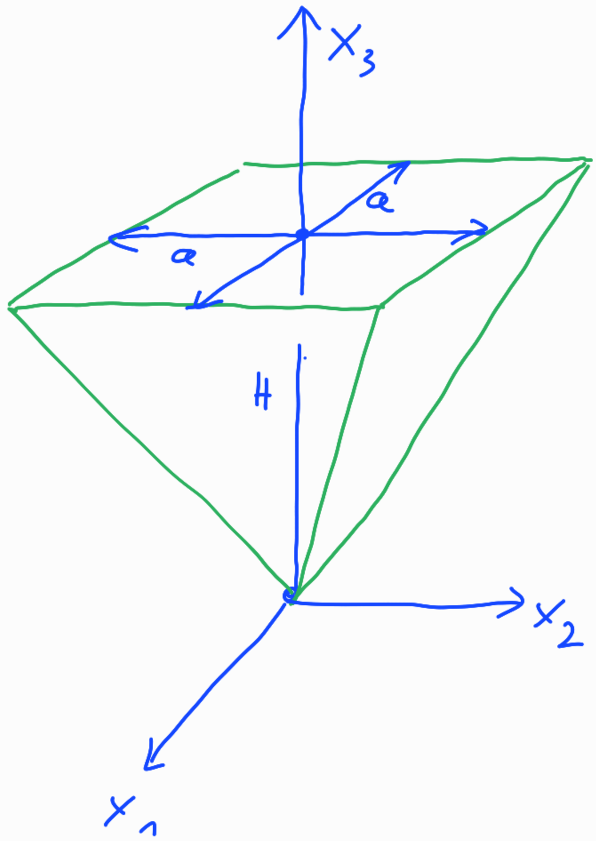
Höhe H



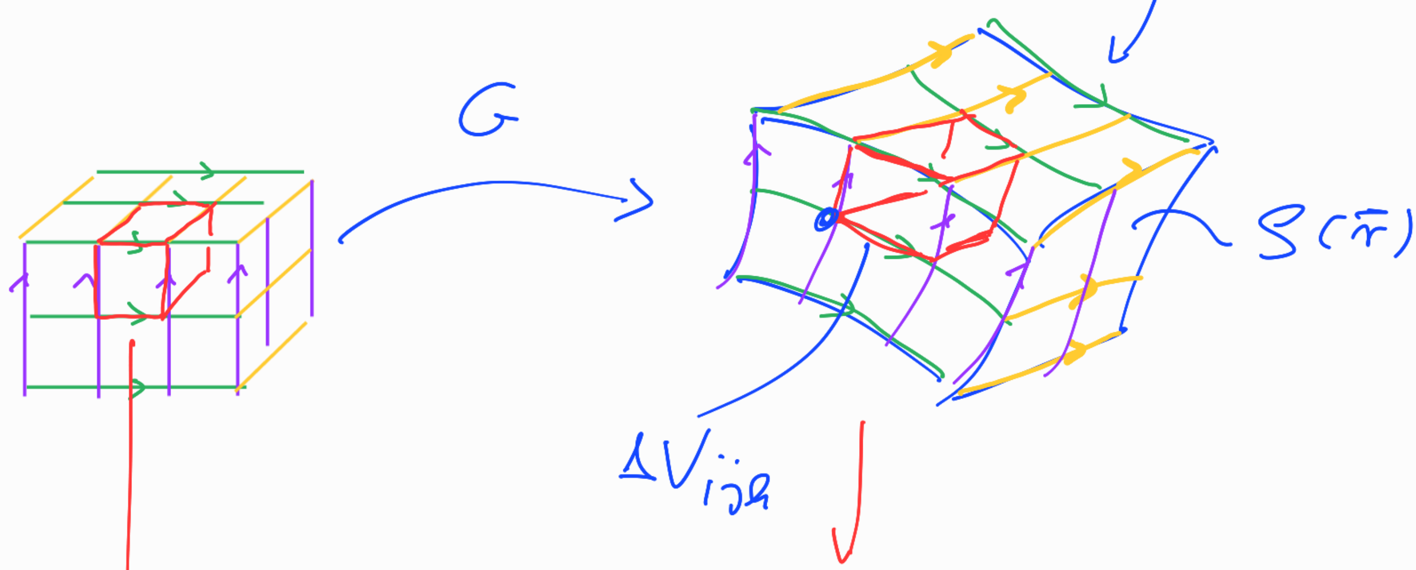
$$P : \left[-\tfrac{a}{2}, \tfrac{a}{2}\right] \times \left[-\tfrac{a}{2}, \tfrac{a}{2}\right] \times [0,H] \to \mathbb{R}^3$$

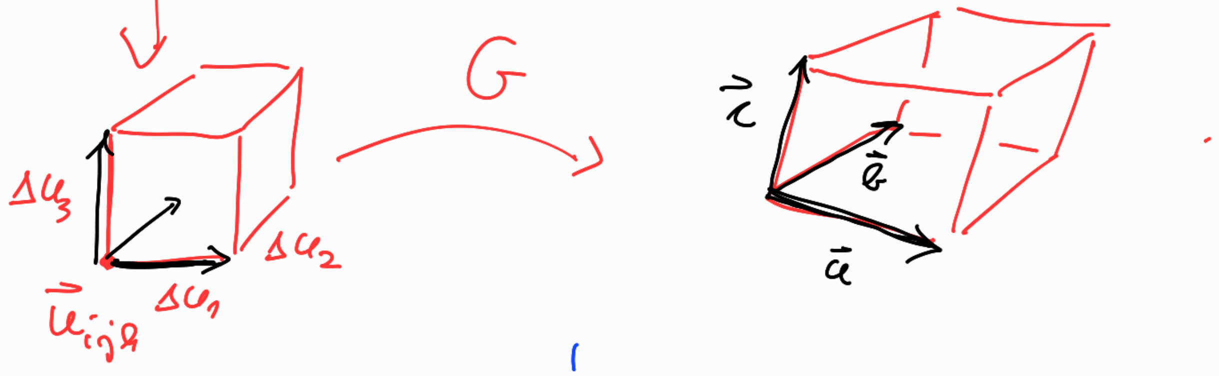
$$(u_1\ ,\ u_2\ ,\ z)$$

$$\mapsto \vec{p}(u_1,u_2,z) = \begin{pmatrix} u_1 \cdot z/H \\ u_2 \cdot z/H \\ z \end{pmatrix}$$



Volumenintegral





$$M(G) = \int_G \rho \, dV \overset{!}{=} \sum_{ijk} \rho_{ijk} \cdot \Delta V_{ijk}$$

$$\Delta V_{ijk} = |\langle \vec{a} \times \vec{b}, \vec{c} \rangle| = |\det(\vec{a}, \vec{b}, \vec{c})|$$

$$\vec{a} = \frac{\partial \vec{G}}{\partial u_1} \cdot \Delta u_1 , \quad \vec{b} = \frac{\partial \vec{G}}{\partial u_2} \Delta u_2 , \quad \vec{c} = \frac{\partial \vec{G}}{\partial u_3} \Delta u_3$$

$$\boxed{\Delta V_{ijk} = \left| \det\left(\frac{\partial \vec{G}}{\partial u_1}, \frac{\partial \vec{G}}{\partial u_2}, \frac{\partial \vec{G}}{\partial u_3} \right) \right| \Delta u_1 \Delta u_2 \Delta u_3}$$

$$\rho_{ijk} = \rho(\vec{G}(\vec{u}_{ijk}))$$

$$\int_G \rho \, dV = \sum_{ijk} \rho(\vec{G}(\vec{u}_{ijk})) \left| \det\left(\frac{\partial \vec{G}}{\partial u_1}, \frac{\partial \vec{G}}{\partial u_2}, \frac{\partial \vec{G}}{\partial u_3} \right) \right| \cdot \Delta u_1 \, \Delta u_2 \, \Delta u_3$$

$$\boxed{\int_G f \, dV := \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(\vec{G}(\vec{u})) \underbrace{\left| \det\left(\frac{\partial \vec{G}}{\partial u_1}, \frac{\partial \vec{G}}{\partial u_2}, \frac{\partial \vec{G}}{\partial u_3} \right) \right| du_1 \, du_2 \, du_3}_{= \, dV}}$$

$(f : \mathbb{R}^3 \to \mathbb{R})$

Volumenelement

<u>Beispiel:</u> Vollzylinder Z

$$Z : [0, 2\pi] \times [0, R] \times [0, H] \longrightarrow \mathbb{R}^3$$

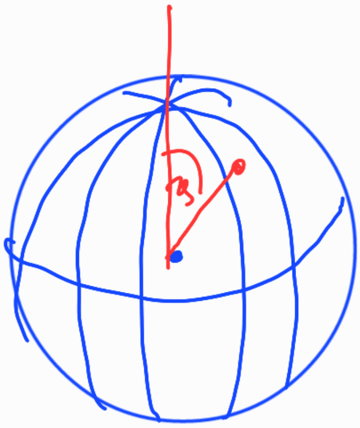
$$(\varphi, \rho, z) \mapsto \vec{Z}(\varphi, \rho, z) = \begin{pmatrix} \rho \cos\varphi \\ \rho \sin\varphi \\ z \end{pmatrix}$$

$$\frac{\partial \vec{Z}}{\partial \varphi} = \begin{pmatrix} -\rho \sin\varphi \\ \rho \cos\varphi \\ 0 \end{pmatrix}, \quad \frac{\partial \vec{Z}}{\partial \rho} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}, \quad \frac{\partial \vec{Z}}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \left| \det \begin{pmatrix} -\rho \sin\varphi & \cos\varphi & 0 \\ \rho \cos\varphi & \sin\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = \left| -\rho \underbrace{(\sin^2\varphi + \cos^2\varphi)}_{=1} \right| = \rho$$

$$\boxed{\int_Z f \, dV = \int_0^{2\pi} \int_0^R \int_0^H f(\varphi, \rho, z) \, \rho \, d\varphi \, d\rho \, dz}$$

Vollkugel



$$K : [0,\pi] \times [0,2\pi] \times [0,R] \mapsto \mathbb{R}^2$$

$$(\vartheta \;,\; \varphi \;,\; r) \longmapsto \underset{\vec{K}(\vartheta,\varphi,r)}{\underset{\parallel}{\vec{r}(\vartheta,\varphi,r)}} = r \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$$

$$\frac{\partial \vec{K}}{\partial \vartheta} = r\, \vec{e}_\vartheta \;,\quad \frac{\partial \vec{K}}{\partial \varphi} = r \sin\vartheta\, \vec{e}_\varphi \;,\quad \frac{\partial \vec{K}}{\partial r} = \vec{e}_r$$

$$\rightarrow \det\left(\frac{\partial \vec{K}}{\partial \vartheta}, \frac{\partial \vec{K}}{\partial \varphi}, \frac{\partial \vec{K}}{\partial r} \right) = \det\left(r\,\vec{e}_\vartheta, r \sin\vartheta\, \vec{e}_\varphi, \vec{e}_r \right)$$

$$r^2 \sin\vartheta \underbrace{\det\left(\vec{e}_\vartheta, \vec{e}_\varphi, \vec{e}_r \right)}_{=1} = r^2 \sin\vartheta$$

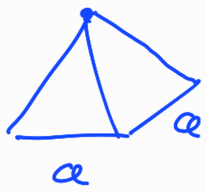
$$\boxed{\int_K f\, dV = \int_0^\pi \int_0^{2\pi} \int_0^R f(\vartheta,\varphi,r) \underbrace{r^2 \sin\vartheta \, d\vartheta\, d\varphi\, dr}_{dV}}$$

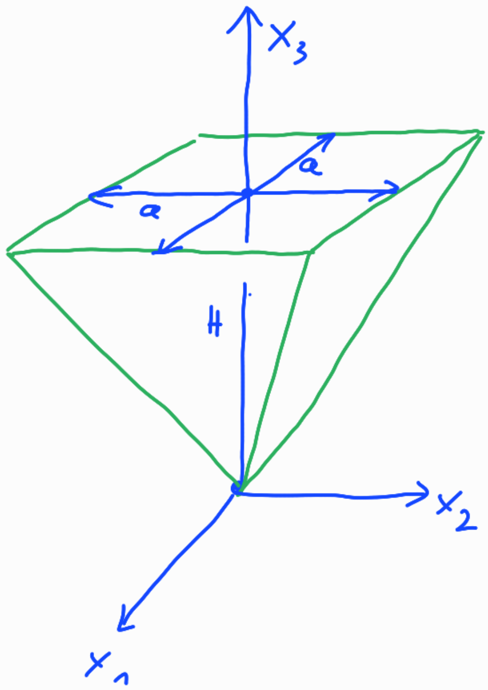
Pyramide:

$$P : \left[-\tfrac{a}{2}, \tfrac{a}{2}\right] \times \left[-\tfrac{a}{2}, \tfrac{a}{2}\right] \times [0, H] \to \mathbb{R}^3$$

$$(u_1, u_2, z) \mapsto \vec{P}(u_1, u_2, z) = \begin{pmatrix} u_1 \cdot z/H \\ u_2 \cdot z/H \\ z \end{pmatrix}$$

Höhe H





$$\downarrow$$

$$\frac{\partial \vec{P}}{\partial u_1} = \begin{pmatrix} z/H \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial \vec{P}}{\partial u_2} = \begin{pmatrix} 0 \\ z/H \\ 0 \end{pmatrix}$$

$$\frac{\partial \vec{P}}{\partial z} = \begin{pmatrix} u_1/H \\ u_2/H \\ 1 \end{pmatrix}$$

$$\boxed{\mathrm{Vol}(G) := \int_G dV}$$

Volumen der Pyramide P: $\mathrm{Vol}(P) =$

$$\to \det\left(\frac{\partial \vec{P}}{\partial u_1}, \frac{\partial \vec{P}}{\partial u_2}, \frac{\partial \vec{P}}{\partial u_3}\right) = \det\begin{pmatrix} z/H & 0 & u_1/H \\ 0 & z/H & u_2/H \\ 0 & 0 & 1 \end{pmatrix} = \underline{\underline{z^2/H^2}}$$

$$\to \mathrm{Vol}(P) = \int_{-a}^{a}\int_{-a}^{a}\int_{0}^{H} 1 \, \frac{z^2}{H^2} \cdot du_1\, du_2\, dz = \underbrace{\int_{-a}^{a} du_1}_{= a} \underbrace{\int_{-a}^{a} du_2}_{= a} \underbrace{\int_{0}^{H} \frac{z^2}{H^2}\, dz}_{H/3}$$

$$= a^2 H/3$$

n beliebig:

$$\int_G f \, d^n V = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(\vec{u}) \det\left(\frac{\partial \vec{G}}{\partial u_1}, \ldots, \frac{\partial \vec{G}}{\partial u_n}\right) du_1 \ldots du_n .$$