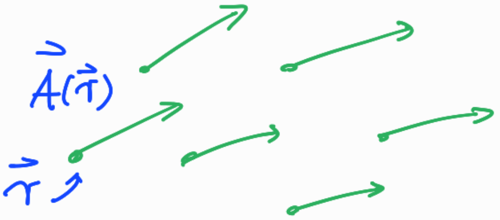
Vektoranalysis

Gradient
Divergenz
Rotation ↙

- Differenzialoperatoren
- Integralsätze von Gauß und Stokes

↳ allg. Vektorfeld:



$$\text{Vf.} \quad \vec{A} : \mathbb{R}^3 \to \mathbb{R}^3$$
$$\vec{r} \mapsto \vec{A}(\vec{r})$$

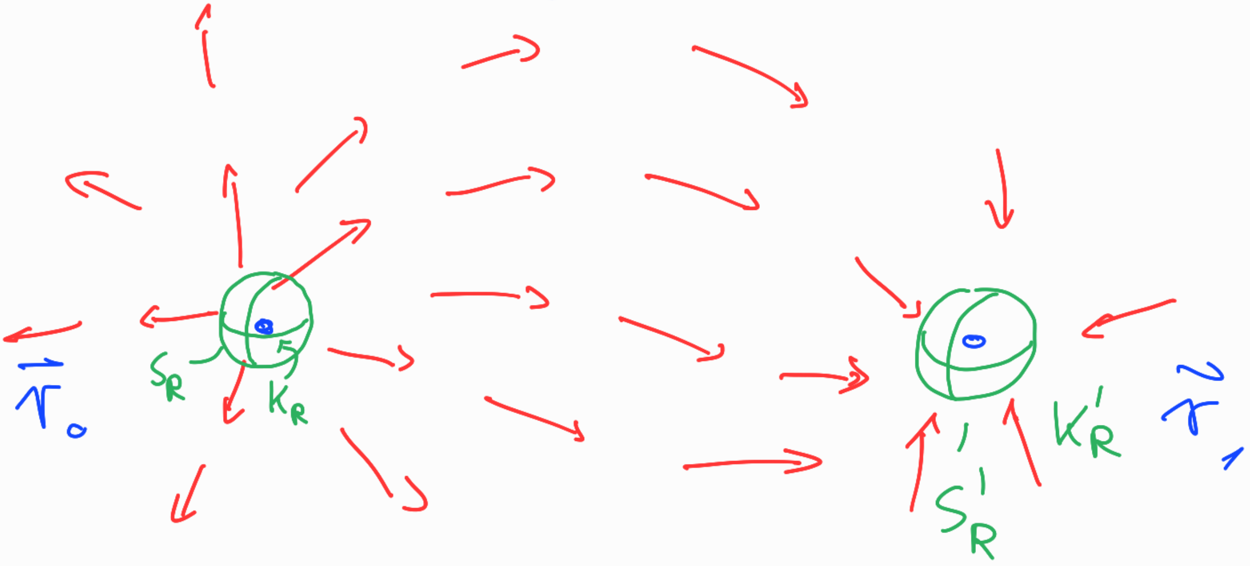
⌐ Bsp.:
- Kraftfeld: $\vec{F}(\vec{r})$
- Stromdichte: $\vec{j}(\vec{r})$
- Elektrische Feld $\vec{E}(\vec{r})$
- Magnetfeld $\vec{B}(\vec{r})$ ⌟

Divergenz eines Vektorfelds $\vec{A}$

↳ = Quellstärke des Vf.s $\vec{A}$: $\operatorname{div} \vec{A}(\vec{r})$

$= ?$

Vf. $\vec{A}$ z.B. Geschwindigkeitsfeld einer stat. Strömung:



Quelle bei $\vec{r}_0$

Senke bei $\vec{r}_1$

$\operatorname{div} \vec{A}(\vec{r}_0) > 0$

$\operatorname{div} \vec{A}(\vec{r}_1) < 0$

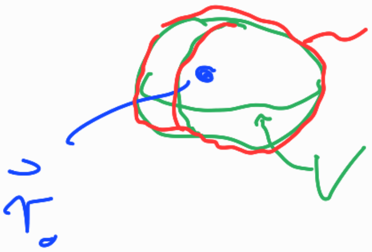
$\int_{S_R} \vec{A} d\vec{f} > 0$

$\int_{S_R'} \vec{A} d\vec{f} < 0$!

$$\rightarrow \operatorname{div} \vec{A}(\vec{r}_0) := \lim_{R \to 0} \frac{1}{Vol(K_R)} \int_{S_R} \vec{A}\, d\vec{f}$$

allg.: **Def:** Divergenz des VF $\vec{A}$ in $\vec{r}_0$:

$$\operatorname{div}\vec{A}(\vec{r}_0) := \lim_{|V|\to 0} \frac{1}{|V|}\int_{\partial V} \vec{A}\, d\vec{f}$$



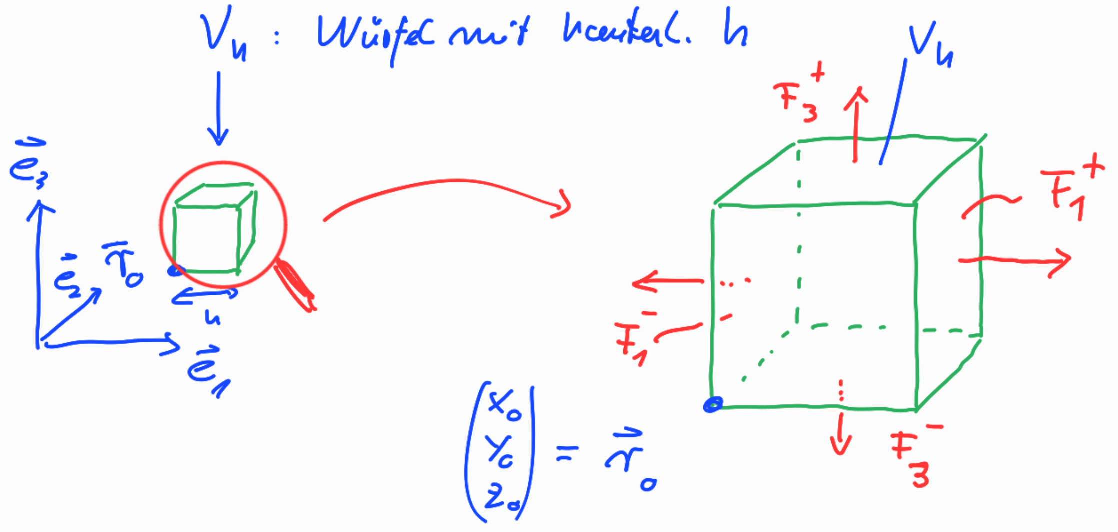
∂V : Oberfläche (Rand) von V

V : Volumengebiet mit $\vec{r}_0 \in V$

und Volumeninhalt $|V|$

Berechnung der Divergenz in kartesischen Koord.

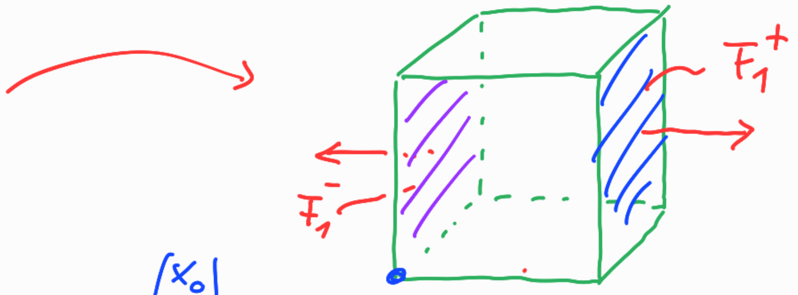
V_h : Würfel mit Kantenl. h

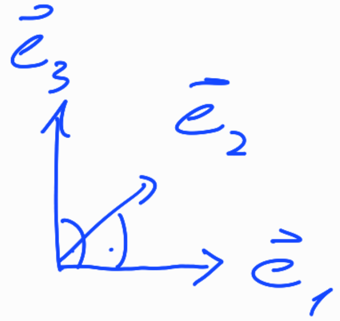


$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \vec{r}_0$$

$$\partial V_h = F_1^+ \cup F_1^- \cup F_2^+ \cup F_2^- \cup F_3^+ \cup F_3^-$$

$$\int_{\partial V_h} \vec{A}\, d\vec{f} = \underbrace{\int_{F_1^+ \cup F_1^-} \vec{A}\, d\vec{f}}_{I_1} + \underbrace{\int_{F_2^+ \cup F_2^-} \vec{A}\, d\vec{f}}_{I_2} + \underbrace{\int_{F_3^+ \cup F_3^-} \vec{A}\, d\vec{f}}_{I_3}$$





$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \vec{r}_0$$

$$I_1 = \int_{y_0}^{y_0+h} \int_{z_0}^{z_0+h} \underbrace{\{A_1(x_0+h,y,z) - A_1(x_0,y,z)\}}_{} \, dy\, dz$$

$$h \ll 1 \longrightarrow \; \|$$

$$\frac{\partial A_1}{\partial x_1}(x_0,y,z) \cdot h$$

$$\rightarrow I_1 = h^3 \frac{\partial A_1}{\partial x_1}(\vec{r}_0) \quad !$$

analog: $I_2 = h^3 \frac{\partial A_2}{\partial x_2}(\vec{r}_0)$, $I_3 = h^3 \frac{\partial A_3}{\partial x_3}(\vec{r}_0)$

$$\hookrightarrow : \underline{\operatorname{div} \vec{A}(\vec{r}_0)} = \frac{1}{h^3} \int_{\partial V_h} \vec{A}\, d\vec{f}$$

$$= \frac{1}{h^3}(I_1 + I_2 + I_3)$$

$$= \underline{\frac{\partial A_1}{\partial x_1}(\vec{r}_0) + \frac{\partial A_2}{\partial x_2}(\vec{r}_0) + \frac{\partial \overrightarrow{A_2}}{\partial x_3}(\vec{r}_0)}$$

Div. in kart. Koordinaten:

$$\boxed{\operatorname{div}\vec{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}}$$

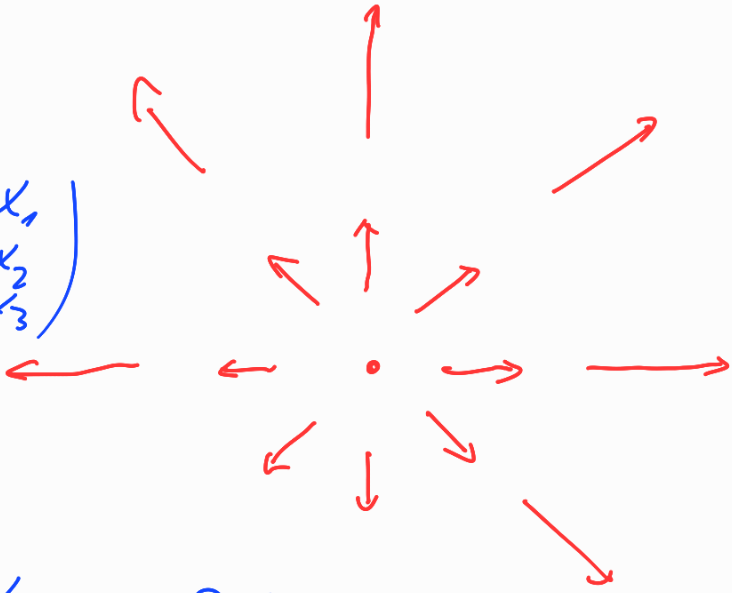
$$= \sum_{i=1}^{3} \frac{\partial A_i}{\partial x_i}$$

$$= \vec{\nabla} \circ \vec{A}$$

$$\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \circ \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

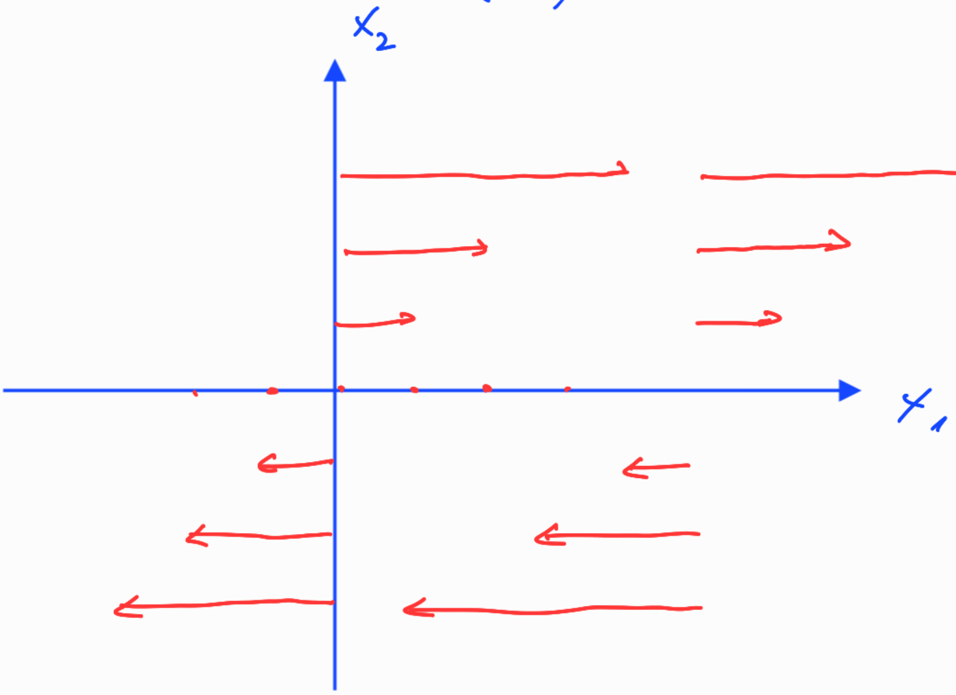
Beispiele:

1) $\vec{A}(\vec{r}) = \vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$



$$\operatorname{div}\vec{A}(\vec{r}) = \underset{\substack{\| \\ 1}}{\frac{\partial x_1}{\partial x_1}} + \underset{\substack{\| \\ 1}}{\frac{\partial x_2}{\partial x_2}} + \underset{\substack{\| \\ 1}}{\frac{\partial x_3}{\partial x_3}} = 3$$

2) $\vec{A}(\vec{r}) = x_2 \vec{e}_1 = \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix}$

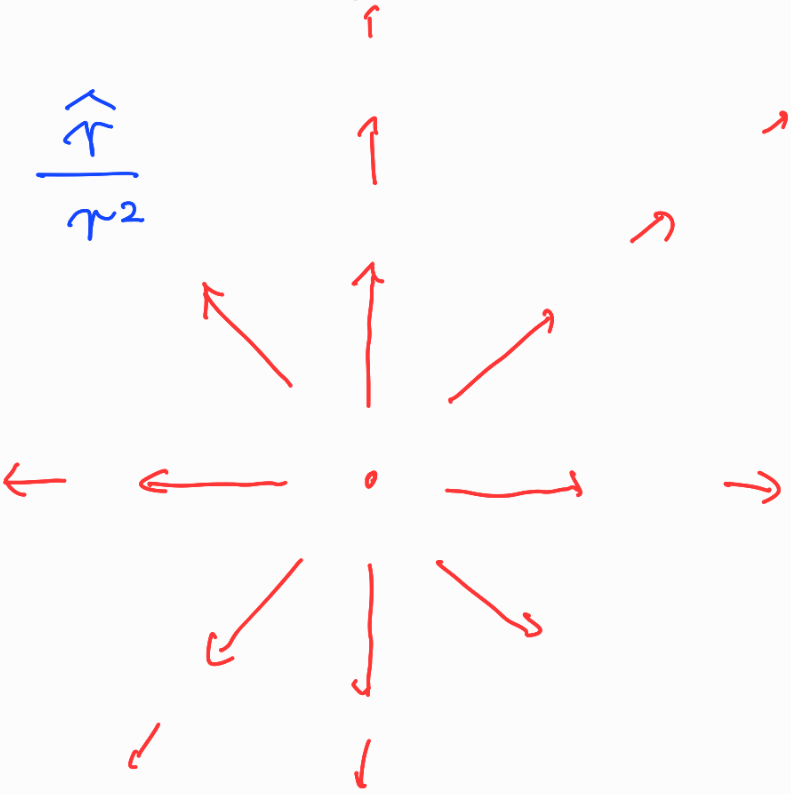


$$\mathrm{div}\, x_2 \vec{e}_1 = \mathrm{div} \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix} = \underset{\underset{0}{\shortparallel}}{\frac{\partial x_2}{\partial x_1}} + 0 + 0 = 0 \, !$$

3) Stromdichte einer Punktquelle in 0:

$$\vec{j}(\vec{r}) = \frac{I_0}{4\pi} \frac{\hat{r}}{r^2}$$

$$\mathrm{div}\, \underset{\underset{0}{\neq}}{\vec{j}(\vec{r})} = 0 \, !$$



$$\mathrm{div}\,\frac{\hat{r}}{r^2} = \mathrm{div}\,\frac{\vec{r}}{r^3} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(x_i \frac{1}{r^3} \right)$$

$\hat{r} = \frac{\vec{r}}{r}$

NR:
$$\frac{\partial}{\partial x_1}\left(x_1 \frac{1}{r^3} \right) = \frac{1}{r^3} + x_1 \frac{\partial}{\partial x_1}\left(\frac{1}{r^3} \right)$$

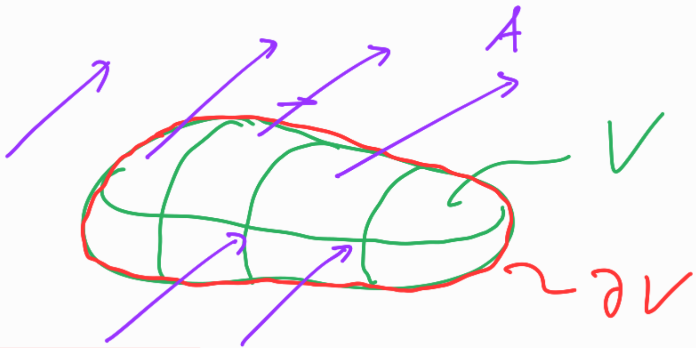
$$= \frac{1}{r^3} + x_1 (-3) \frac{1}{r^4} \underbrace{\frac{\partial}{\partial x_1} \left(x_1^2 + x_2^2 + x_3^2 \right)^{1/2}}_{\frac{1}{\cancel{2} r} \cancel{2} x_1}$$

$$= \frac{1}{r^3} - 3 \frac{x_1^2}{r^5}$$

$$\mathrm{div}\,\frac{\hat{r}}{r^2} = \sum_{i=1}^{3} \left(\frac{1}{r^3} - 3 \frac{x_i^2}{r^5} \right)$$

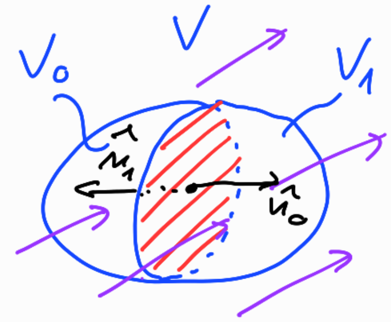
$$= \frac{3}{r^3} - \frac{3}{r^5} \underbrace{\sum_i x_i^2}_{r^2} = 0 \;!$$

Satz von Gauß



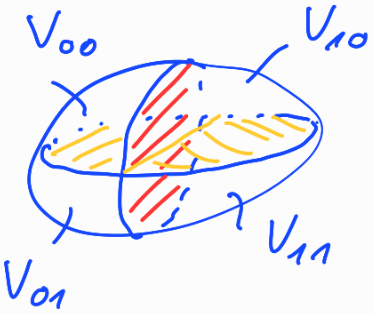
$$\boxed{\int_{\partial V} \vec{A}\, d\vec{f} \overset{!}{=} \int_{V} \operatorname{div}\vec{A}\; dV}$$

"Physikerbeweis":



$$\int_{\partial V} \vec{A}\, d\vec{f} \overset{!}{=} \int_{\partial V_0} \vec{A}\, d\vec{f} + \int_{\partial V_1} \vec{A}\, d\vec{f}$$

$$\overset{!}{=} \int_{\partial V_{00}} \vec{A}\, d\vec{f} + \int_{\partial V_{01}} \vec{A}\, d\vec{f} + \int_{\partial V_{10}} \vec{A}\, d\vec{f} + \int_{\partial V_{11}} \vec{A}\, d\vec{f}$$



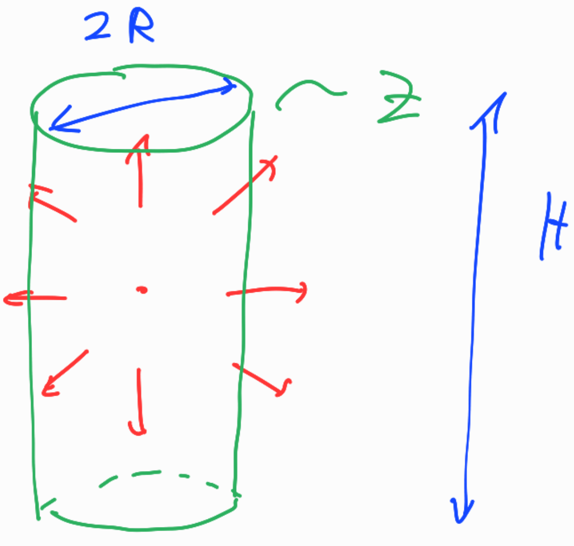
$\vdots$

$$= \underset{\downarrow}{\sum_i} \frac{1}{|V_i|} \int_{\partial V_i} \vec{A}\, d\vec{f}\; \underset{\downarrow}{|V_i|}$$

$$= \int_V \operatorname{div}\vec{A}\;\; dV$$

Anwendungsbeispiele:

1) $\vec{A}(\vec{r}) = \vec{r} \;,\rightarrow\; \operatorname{div}\vec{A} = 3$

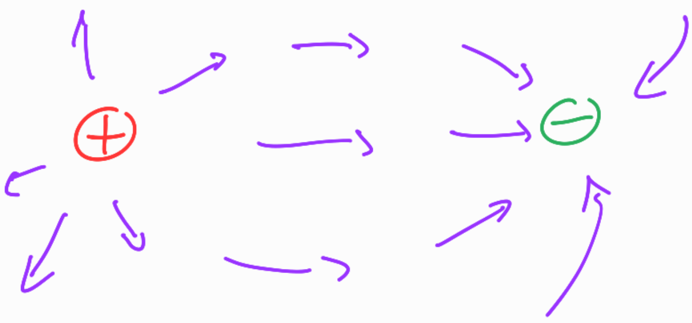


$$? = \int_{\partial Z} \vec{A}\, d\vec{f} \overset{\text{S.v.G.}}{=} \int_{Z} \operatorname{div}\vec{A}\, dV$$

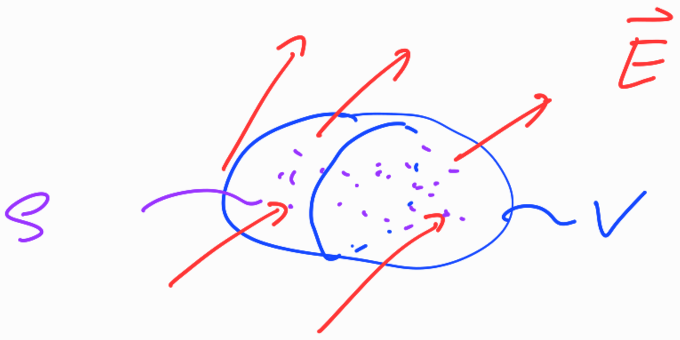
$$= 3 \int_{Z} dV = 3 \cdot \pi R^2 \cdot H \quad !$$

Elektrostatik:

„elekt. Ladungen sind die Quellen des elek. Feldes!"



$$\boxed{\operatorname{div}\vec{E} \overset{!}{=} \rho/\varepsilon_0}$$



$$\boxed{\operatorname{div}\vec{E} = \underline{\underline{\rho}}/\varepsilon_0}$$

$$Q_V = \int_V \underline{\underline{\rho}}\, dV = \varepsilon_0 \int_V \operatorname{div}\vec{E}\, dV$$

elek. Ldg in V

$$\underset{\text{s.v.G.}}{=} \varepsilon_0 \int_{\partial V} \vec{E}\, dV$$

d.h.

$$\boxed{\int_{\partial V} \vec{E}\, d\vec{f} = \frac{1}{\varepsilon_0} Q_V}$$