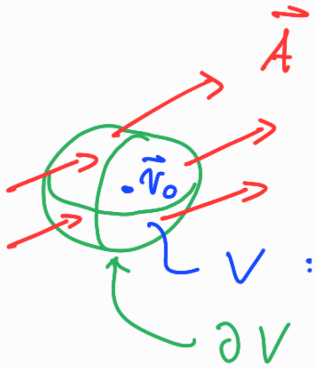
Wiederholung:

Divergenz („Quellstärke") eines Vfs:



$$\boxed{\operatorname{div} A(\vec{r}_0) = \lim_{|V|\to 0} \frac{1}{|V|} \int_{\partial V} \vec{A}\, d\vec{f}}$$

V : Volumengebiet um $\vec{r}_0$,

∂V : Rand(oberfläche) von V

$|V|$: Volumeninhalt

Berechnung in kartesischen Koordinaten:

$$\operatorname{div} \vec{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

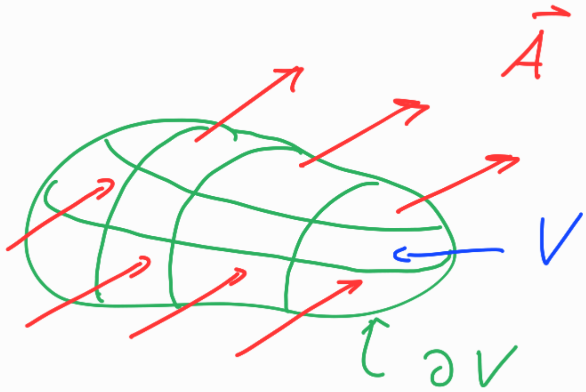
$$= \vec{\nabla} \cdot \vec{A}$$

$$\vec{\nabla} = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix}, \quad \vec{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Bsp.:

- $\operatorname{div} \vec{r} = 3$
- $\operatorname{div} \dfrac{\hat{r}}{r^2} = 0 \quad (\text{für } r \neq 0)$

Satz von Gauß



$$\boxed{\int_{\partial V} \vec{A}\, d\vec{f} \overset{!}{=} \int_{V} \operatorname{div} \vec{A}\, dV}$$

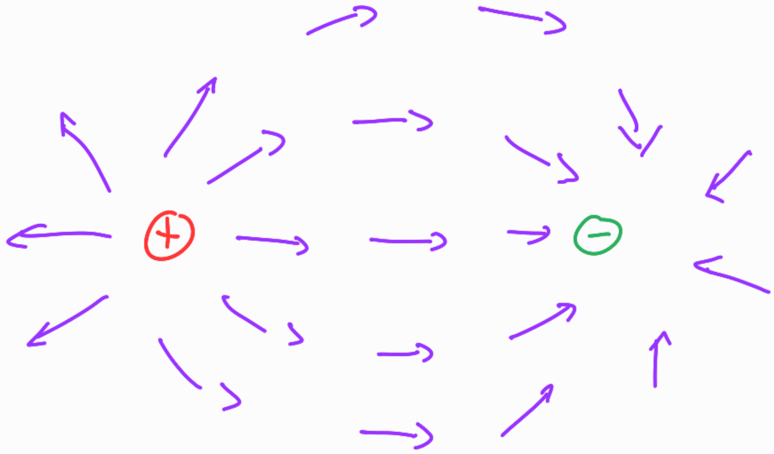
Fluss von $\vec{A}$ durch Oberfl. von V

Gesamtquellstärke in V

Anwendung von Divergenz und S.v.G. in der Elektrostatik:

„(negative) elektr. Ladungen sind Quellen (Senken) des elektrischen Feldes"

$\vec{E}$



<u>elektrisches Feld:</u> $\vec{E}(r) = \vec{F}/q$

Kraft auf Punktladung q am Ort $\vec{r}$

<u>Ladungsdichte</u> $\rho(\vec{r}) = \dfrac{\Delta Q(\vec{r})}{\Delta V}$

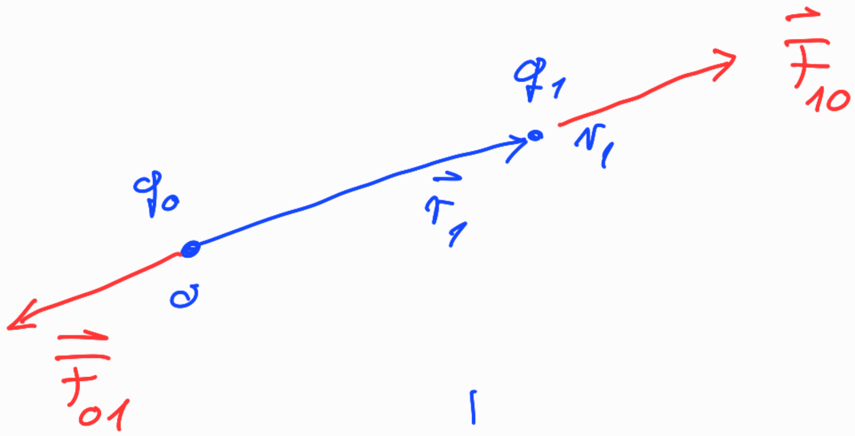
Gaußsche Gesetz:

$$\boxed{\operatorname{div}\vec{E} = \rho/\varepsilon_0}\quad !$$

$$\varepsilon_0 = 8{,}85\cdot 10^{-12}\,\frac{As}{Vm}$$

→ el. Kraft zwisch. elektr. geladen Körpern!

→ z.B. Kraft zwischen Punktladungen:



Coulomb: $$\vec{F}_{10} \overset{!}{=} \frac{1}{4\pi\varepsilon_0}\,\frac{q_0 q_1}{r_1^2}\,\hat{r}_1$$

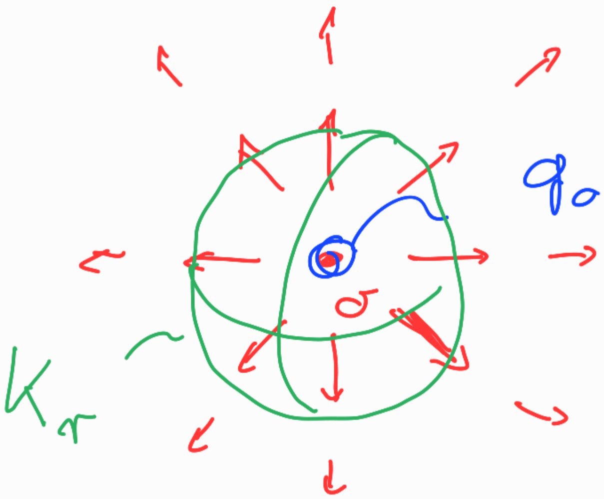
$q_0 \longrightarrow$ elektrischen Feld $\vec{E}(\vec{r}) = ?$

in 0 $\longrightarrow$ Kraft auf Pkt.ladg q_1:

$$\vec{F}_{10} = \vec{E}(\vec{r}_1)\, q_1$$

elektr. Feld einer Pktldg q_0 in σ:

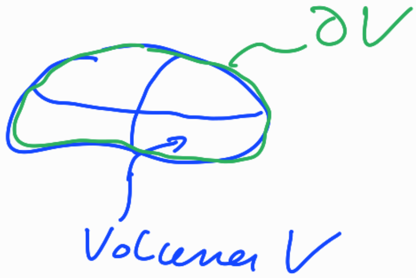
Symmetrie: → Ansatz:



$$\vec{E}(\vec{r}) = \underbrace{E(r)}_{\text{el. Feldstärke}}\,\hat{r}$$

Bestimmung der Feldst. $E(r)$:

$$\operatorname{div}\vec{E} = \rho/\varepsilon_0 \quad (*)$$



$$\int_{\partial V} \vec{E}\,d\vec{f} \overset{\text{S. v. G.}}{=} \int_V \operatorname{div}\vec{E}\,dV \overset{(*)}{=} \frac{1}{\varepsilon_0}\underbrace{\int_V \rho\,dV}_{Q_V}$$

$$\boxed{\int_{\partial V} \vec{E}\,d\vec{f} = \frac{Q_V}{\varepsilon_0}}$$

Gaußsches Gesetz in integraler Form

für Wahl $V = K_r$:

$\hookrightarrow \partial V = \partial K_r = S_r$

$$\hookrightarrow \underbrace{\oint_{S_r} \vec{E}\, d\vec{f}} \overset{!}{=} \frac{Q_{K_r}}{\varepsilon_0} = q_0/\varepsilon_0$$

pkt. ldg q_0 in 0

$$\langle \vec{E}, d\vec{f} \rangle \overset{!}{=} E(r)\, |d\vec{f}|$$

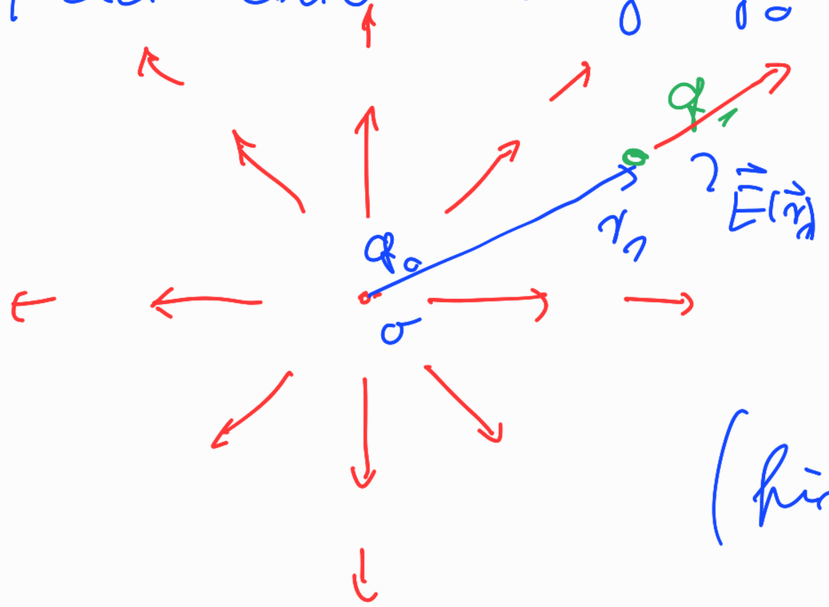
$$\vec{E} = E(r)\, \hat{r}$$

$$\oint_{S_r} \vec{E}\, d\vec{f} = \oint_{S_r} E(r)\, |d\vec{f}| = E(r) \underbrace{\oint_{S_r} |d\vec{f}|}_{4\pi r^2}$$

d.h. $\quad 4\pi r^2 E(r) \overset{!}{=} q_0/\varepsilon_0$

$$\rightarrow E(r) = \frac{1}{4\pi\varepsilon_0} \frac{q_0}{r^2} \quad \hookrightarrow \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q_0}{r^2} \hat{r}$$

Feld einer Punktladung q_0 in O:



$$\vec{E}(\vec{r}) \overset{!}{=} \frac{1}{4\pi\varepsilon_0} \frac{q_0}{r^2} \hat{r} \qquad (*)$$

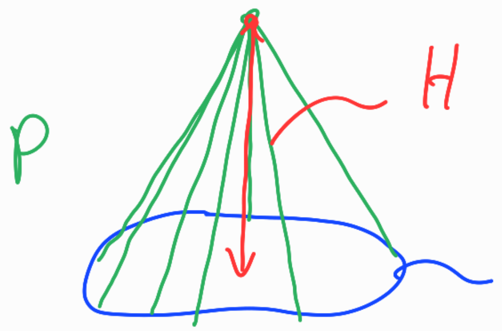
$$\left(\text{für } \vec{r} \neq 0: \quad \operatorname{div} \vec{E} \overset{!}{=} 0 \right)$$

$$\rightarrow \quad \vec{F}_{10} = q_1 \vec{E}(\vec{r}_1) \overset{(*)}{=} \frac{1}{4\pi\varepsilon_0} \frac{q_0 q_1}{r_1^2} \hat{r}_1$$

2. Anwendung des S. v. Gauß:

<u>Geometrie</u>: allg. Pyramide:

$$V = \frac{a^2 H}{3}$$



Grundfläche G

Flächeninhalt $|G|$

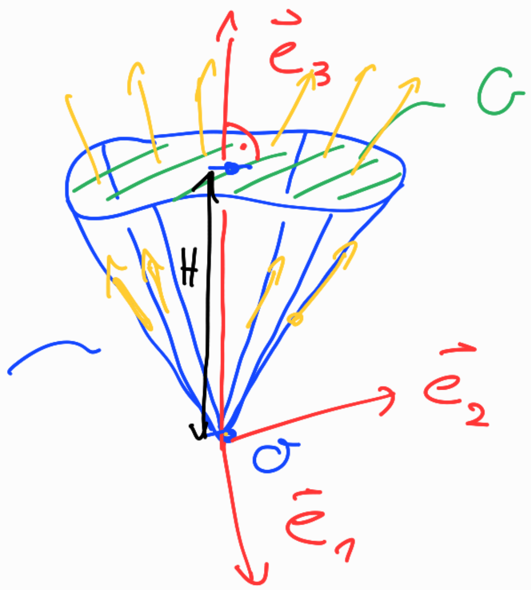
$$V = \frac{\pi R^2 H}{3}$$

Volumen der Pyramide P:

$$\boxed{V = \frac{|G| \cdot H}{3}}$$

$\rightsquigarrow$ Beweis mit S. von Gauß!

1)



2) Vf. $\vec{A}(\vec{r}) = \vec{r}$! $\rightarrow$

$\operatorname{div} \vec{A} = \operatorname{div} \vec{r} = 3$ konstant!

$$\int_P \operatorname{div} \vec{A} \, dV = \int_P 3 \, dV = 3 \int_P dV = 3V$$

$$\underset{\text{s.v. Gauß}}{=} \int_{\partial P} \vec{A} \, d\vec{f} = \int_G H \, |d\vec{f}| = H \, |G|$$

$$\boxed{V = \frac{|G| H}{3}}$$

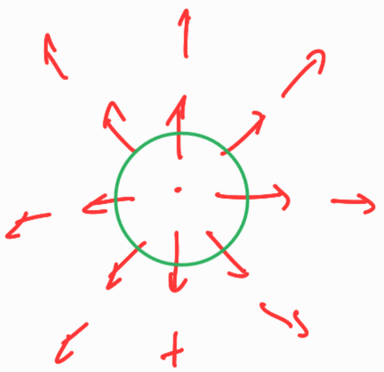
- kein Beitrag vom Mantel der Pyramide (da dort $\vec{A} = \vec{r} \parallel \hat{n}$!)
- $\vec{r} \in G$: $\vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ H \end{pmatrix}$!

$\rightarrow \langle \vec{A}, d\vec{f} \rangle = H \, |d\vec{f}|$

$\vec{A} = \vec{r}$, $d\vec{f} = \vec{e}_3 |d\vec{f}|$

Rotation (Wirbelstärke) eines Vfs.

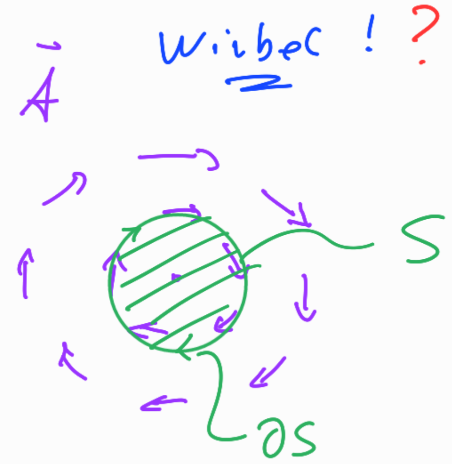
wirbelfreies Vf: !



$$\int_{\partial S} \vec A \, d\vec l = 0 \quad !$$

Rotation = 0

Wirbel! ?



$$\int_{\partial S} \vec A \, d\vec l \;\overset{>}{\neq}\; 0 \quad !$$

$\hat{=}$ Rotation $\overset{>}{\neq} 0$

Def.: Rotation (Wirbelstärke) des Vfs $\vec A$ im Punkt $\vec r_0$

:= Vektor $\operatorname{rot} \vec A(\vec r_0)$ bestimmt

durch

$$\boxed{\langle \hat n, \operatorname{rot} \vec A(\vec r_0) \rangle := \lim_{|S| \to 0} \frac{1}{|S|} \int_{\partial S} \vec A \, d\vec l}$$

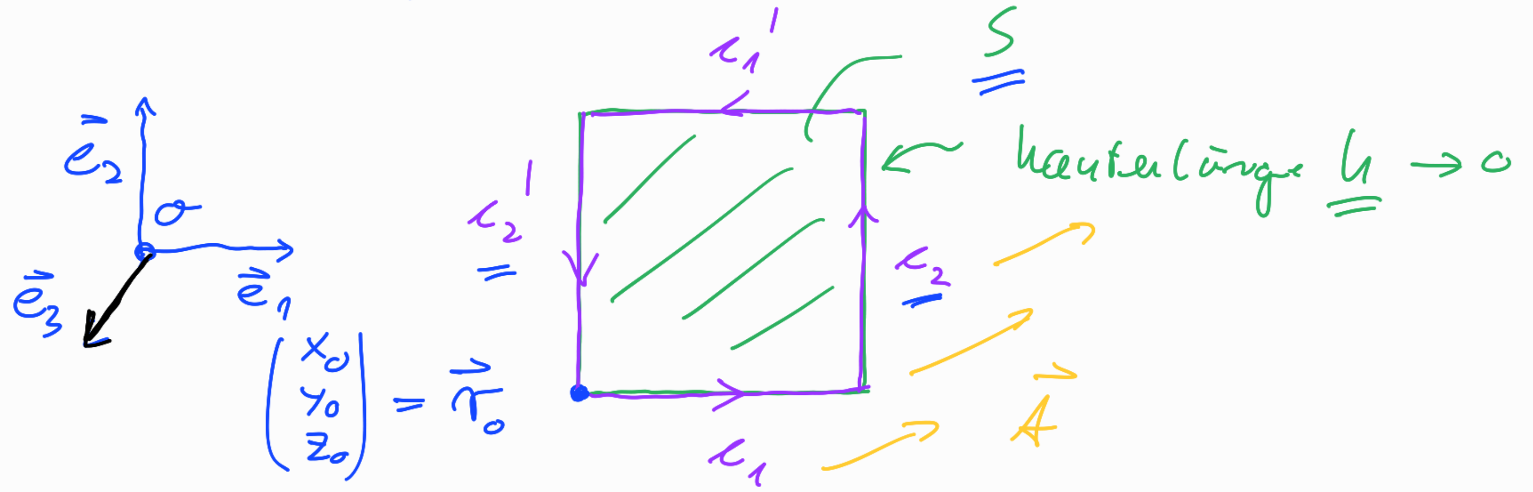
S : Fläche

$\hat n$: Flächennormale

$|S|$: Flächeninhalt

∂S : Flächenrand

Bestimmung der Rotation in kart. Koord.:



$$\partial S = c_1 + c_2 + c_1' + c_2'$$

$$(\operatorname{rot} \vec{A}(\vec{r}_0))_3 = \langle \vec{e}_3, \operatorname{rot} \vec{A}(\vec{r}) \rangle$$

$h \to 0$

$$= \frac{1}{h^2} \int_{x_0}^{x_0+h} \underbrace{\{A_1(x, y_0, z_0) - A_1(x, y_0+h, z_0)\}}_{= -h \frac{\partial A_1}{\partial x_2}(x, y_0, z_0)} dx$$

$$\frac{1}{h^2} \int_{y_0}^{y_0+h} \underbrace{\{A_2(x_0+h, y, z_0) - A_2(x_0, y, z_0)\}}_{= h \frac{\partial A_2}{\partial x_1}(x_0, y, z_0)} dy$$

$h \to 0$

$$= \frac{1}{\cancel{h^2}} \cdot \left(-\cancel{h^2} \frac{\partial A_1}{\partial x_2}(\vec{r}_0)\right) + \frac{\cancel{h^2}}{\cancel{h^2}} \frac{\partial A_2}{\partial x_1}(\vec{r}_0)$$

$$= -\frac{\partial A_1}{\partial x_2}(\vec{r}_0) + \frac{\partial A_2}{\partial x_1}(\vec{r}_0) \overset{!}{=} (\operatorname{rot} \vec{A}(\vec{r}_0))_3$$

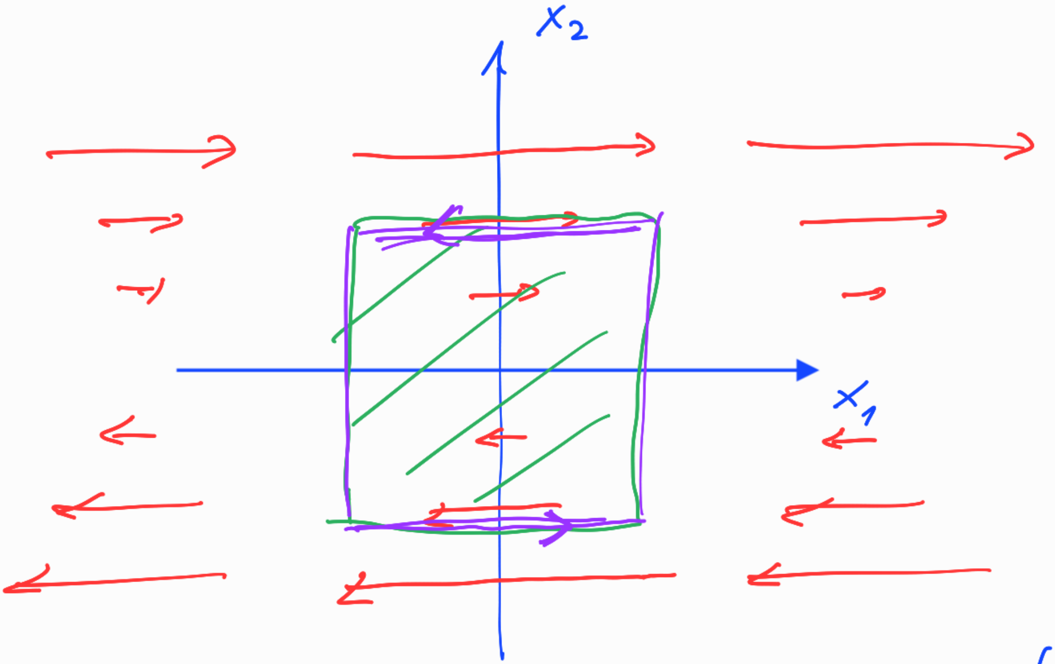
d.h.

$$\boxed{\begin{aligned} (\mathrm{rot}\,\vec{A})_1 &= \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \\ (\mathrm{rot}\,\vec{A})_2 &= \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \\ (\mathrm{rot}\,\vec{A})_3 &= \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \\ \mathrm{rot}\,\vec{A} &= \vec{\nabla} \times \vec{A} \end{aligned}} \qquad (*)$$

$$= \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix} \times \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

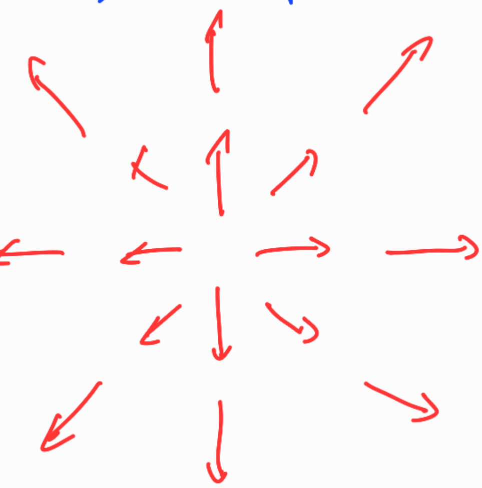
Beispiele:

1) $\vec{A}(\vec{r}) = x_2\,\vec{e}_1 = \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix}$!



$$\mathrm{rot}\begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix} = \vec{\nabla} \times \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} !$$

2) $\vec{f}(\vec{r}) = \vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$



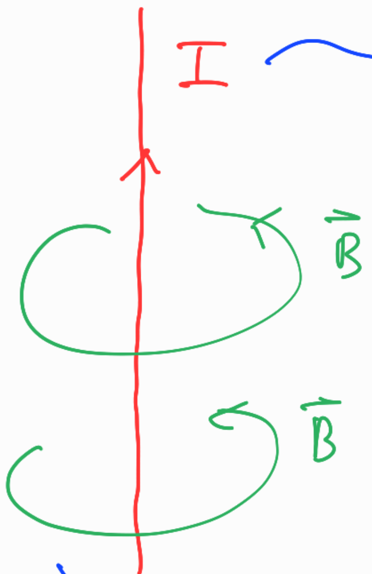
$$\mathrm{rot}\,\vec{r} = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0} \,.$$

Anwendung:

Magnetostatik:

„elektrische Ströme verursachen magnetische Wirbelfelder!"



$I \longrightarrow$ elekt. Stromdichte

$\vec{j}(\vec{r}) \longrightarrow \vec{B}(\vec{r})$

Magnetfeld!

Ampèresche Gesetz: $\boxed{\mathrm{rot}\,\vec{B} \overset{!}{=} \mu_0 \vec{j}}$