gestern:

euklidischer Vektorraum = Vektorraum mit

Skalarprodukt: $V \times V \longrightarrow \mathbb{R}$

$\vec{a}, \vec{b} \longmapsto \langle \vec{a}, \vec{b} \rangle$

mit Eigenschaften

(SP1) $\langle \vec{a}, \vec{b} \rangle = \langle \vec{b}, \vec{a} \rangle$ (Symmetrie)

(SP2) $\langle \vec{a}, \vec{a} \rangle > 0$ für $\vec{a} \neq 0$

" " $= 0$ für $\vec{a} = \vec{0}$ (Positivität)

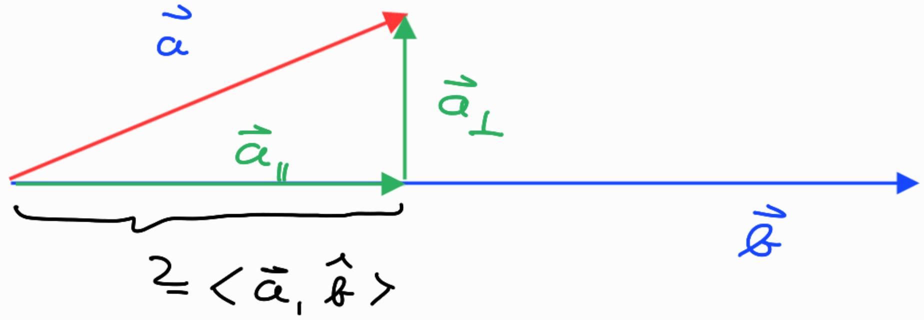
(SP3) $\langle \vec{a}, \vec{b} + \vec{c} \rangle = \langle \vec{a}, \vec{b} \rangle + \langle \vec{a}, \vec{c} \rangle$

$\langle \vec{a}, \lambda \vec{b} \rangle = \lambda \langle \vec{a}, \vec{b} \rangle$ (Linearität)

$\Rightarrow$ Norm $|\vec{a}| := \langle \vec{a}, \vec{a} \rangle^{1/2}$

Orthogonalität

$\vec{a} \perp \vec{b} \quad :\Leftrightarrow \quad \langle \vec{a}, \vec{b} \rangle = 0$



• $\vec a_\parallel = \langle \vec a, \hat b \rangle \, \hat b \quad , \quad \hat b = \vec b / |\vec b|$

$\rightsquigarrow$ $\langle \vec a, \hat b \rangle$ = „Länge der Projektion von $\vec a$ auf $\vec b$"

$\Rightarrow$ • <u>Winkel</u> $\vartheta = \sphericalangle(\vec a, \vec b)$:

$$\cos\vartheta = \langle \hat a, \hat b \rangle = \frac{\langle \vec a, \vec b \rangle}{|\vec a|\,|\vec b|}$$

$$\Rightarrow \quad \langle \vec a, \vec b \rangle = |\vec a|\,|\vec b| \cdot \cos\vartheta$$

• wegen $|\vec a_\parallel| \leq |\vec a|$ (<u>geometr.</u>)

d.h. $|\langle \vec a, \hat b \rangle| \leq |\vec a|$

$$\xrightarrow{\cdot|\vec b|} \quad \boxed{|\langle \vec a, \vec b \rangle| \leq |\vec a|\,|\vec b|}$$

<u>Cauchy-Schwartz-Ungleichung</u>

Orthonormalbasis (ONB)

Basis $B = (\vec{e}_1, \dots, \vec{e}_n)$ Orthonormalbasis

g.d.w. $\vec{e}_i$ normiert und paarweise

orthogonal, d.h.

- $|\vec{e}_i| = 1$
- $\vec{e}_i \perp \vec{e}_j$

d.h.
$$\langle \vec{e}_i, \vec{e}_j \rangle = \begin{cases} 1 & : i = j \\ 0 & : i \neq j \end{cases}$$

mit Kronecker-Symbol $\delta_{ij} := \begin{cases} 1 & : i = j \\ 0 & : i \neq j \end{cases}$:

$$\boxed{\langle \vec{e}_i, \vec{e}_j \rangle = \delta_{ij}}$$

Berechnung des S.P. in Komponenten bzg.

einer ONB: $B = (\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n)$

$$\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_B = \sum_{i=1}^{n} a_i \vec{e}_i$$

$$\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}_B = \sum_{j=1}^{n} b_j \vec{e}_j$$

$$\boxed{\begin{aligned} \langle \vec{a}, \vec{b} \rangle &\overset{!}{=} \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \dots \\ |\vec{a}| &= \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2} \end{aligned}}$$

warum?

$$\langle \vec{a}, \vec{b} \rangle = \left\langle \sum_i a_i \vec{e}_i , \sum_j b_j \vec{e}_j \right\rangle$$

$$= \sum_i \sum_j \langle a_i \vec{e}_i , b_j \vec{e}_j \rangle$$

$$= \sum_{ij} a_i b_j \underbrace{\langle \vec{e}_i , \vec{e}_j \rangle}_{= \delta_{ij}}$$

$$= \sum_{ij} a_i b_j \delta_{ij} = \sum_i a_i b_i$$

Bsp. $\mathbb{R}^n$ mit Standardskalarprodukt

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\langle \vec{x}, \vec{y} \rangle := \sum_{i=1}^{n} x_i y_i \qquad \checkmark$$

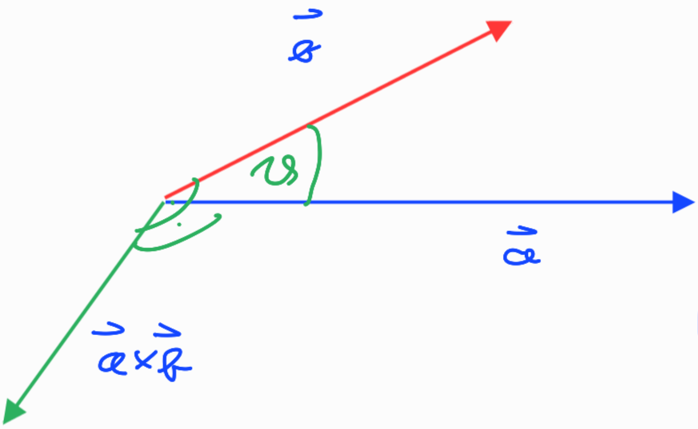
(erfüllt (SP1 − 3))

Standardbasis $B = (\vec{e}_1, \ldots, \vec{e}_n)$ ist ONB:

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Vektorprodukt (Kreuzprodukt) für 3-dim. eukl. VR

geom.:



$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,|\sin \vartheta|$$

Def: Vektorprodukt (Kreuzprodukt) = Abb.

$$V \times V \longrightarrow V$$

$$\vec{a}, \vec{b} \longmapsto \vec{a} \times \vec{b}$$

def. durch

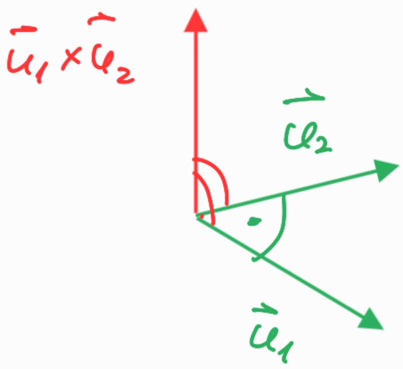
(V1) $\quad \vec{a} \times \vec{b} = - \vec{b} \times \vec{a} \quad$ (Antisymmetrie)

(V2) $\quad \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$\quad \vec{a} \times (\lambda \vec{b}) = \lambda \, (\vec{a} \times \vec{b}) \quad$ (Linearität)

(V3) $\quad \vec{u}_1, \vec{u}_2$ orthonormal

$\Rightarrow \quad \vec{u}_1, \vec{u}_2, \vec{u}_1 \times \vec{u}_2$ rechtshändige ONB

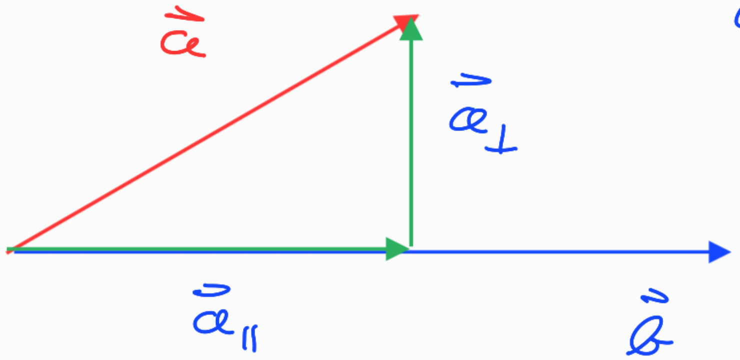


Eigenschaften u. Anwendungen

1) $\vec{a} \times \vec{a} = \vec{0}$

2) $\vec{a} \parallel \vec{b} \iff \vec{a} \times \vec{b} = \vec{0}$

3) Orthogonalkomponente



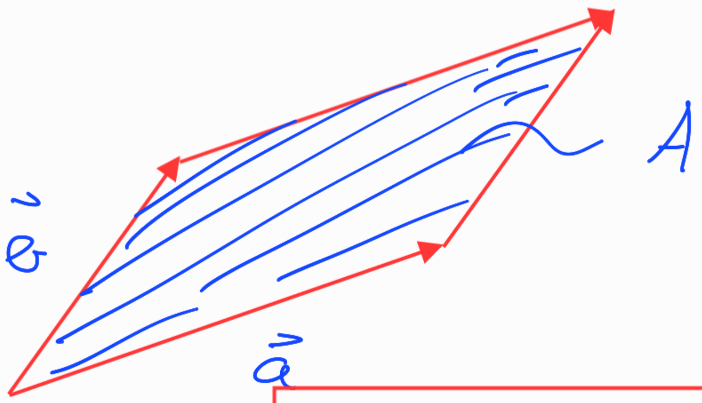
$$\vec{a} = \vec{a}_\parallel + \vec{a}_\perp$$

$$\vec{a}_\parallel = \langle \hat{b}, \vec{a} \rangle \hat{b}$$

$$\boxed{\begin{aligned} |\vec{a}_\perp| &= |\hat{b} \times \vec{a}| \\ \vec{a}_\perp &= (\hat{b} \times \vec{a}) \times \hat{b} \end{aligned}}$$

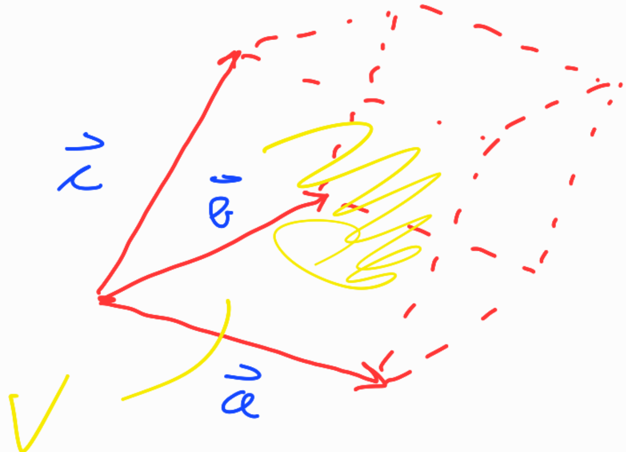
4) $|\vec{a} \times \vec{b}| = |\vec{a}| \, |\vec{b}| \, |\sin \vartheta|$

5) Fläche eines Parallelogramms



Flächeninhalt $\boxed{A = |\vec{a} \times \vec{b}|}$

6) <u>Volumen eines Spats</u> (Parallelepipeds)



Volumeninhalt

$$\boxed{V = |\langle \vec{a} \times \vec{b}, \vec{c} \rangle|}$$

"Spatvolumen"

7) Berechnung des VPs in Komp. bzg. $\underline{\underline{\text{ONB}}}$ $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_B, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_B$$

$$\rightarrow \quad \vec{a} \times \vec{b} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} = \begin{pmatrix} \underline{a_2 b_3 - a_3 b_2} \\ \underline{a_3 b_1 - a_1 b_3} \\ \underline{a_1 b_2 - a_2 b_1} \end{pmatrix}_B$$

Warum?

zu 1): $\quad \vec{a} \times \vec{a} \overset{!}{=} 0$

Antisymmetrie: $\quad \vec{a} \times \vec{a} \overset{!}{=} - \vec{a} \times \vec{a}$

$$\rightarrow \quad \vec{a} \times \vec{a} = 0 \qquad !$$

zu 2): $\quad \vec{a} \parallel \vec{b} \iff \vec{a} \times \vec{b} = \vec{0}$

„$\Rightarrow$" $\quad \vec{b} = \lambda \vec{a} \rightarrow \vec{a} \times \vec{b} = \vec{a} \times \lambda \vec{a}$

$$= \lambda (\vec{a} \times \vec{a}) \overset{1)}{=} \vec{0}$$

„$\Leftarrow$" $\quad$ z.z.: $\quad \vec{a} \nparallel \vec{b} \Rightarrow \vec{a} \times \vec{b} \neq \vec{0} \quad !$

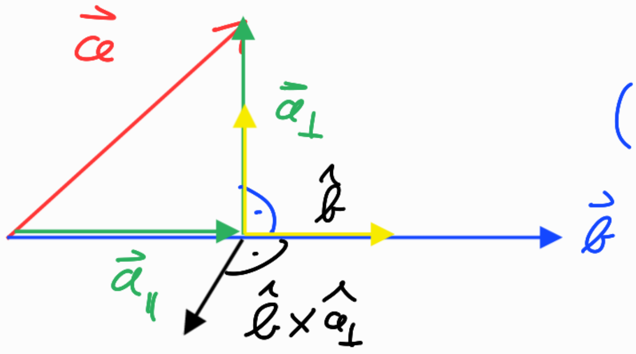
$\vec a \nparallel \vec b$: $\quad \vec a = \vec a_\parallel + \vec a_\perp \quad ; \quad \vec a_\perp \neq \vec 0$!

$$\rightsquigarrow \quad \vec a \times \vec b = (\vec a_\parallel + \vec a_\perp) \times \vec b$$

$$= \underbrace{\vec a_\parallel \times \vec b}_{=\,\vec 0} + \underbrace{\vec a_\perp \times \vec b}_{=\,|\vec a_\perp|\,|\vec b|\,\underbrace{\hat a_\perp \times \hat b}} \neq 0$$

normierter Vektor (nach V3)
$\neq 0$!

zu 3) <u>Orthogonalkomponente</u>



$$|\hat b \times \vec a| = |\vec a_\perp| \quad ✓$$

$$(\hat b \times \vec a) \times \hat b = |\vec a_\perp|\, \hat a_\perp$$

$$= \vec a_\perp \quad ✓$$

$$\vec a = \vec a_\parallel + \vec a_\perp$$

$$\hat b \times \vec a \overset{\downarrow}{=} \hat b \times (\vec a_\parallel + \vec a_\perp) = \overbrace{\hat b \times \vec a_\parallel}^{\vec 0} + \hat b \times \vec a_\perp$$

$$= |\vec a_\perp| \; \underbrace{\hat b \times \hat a_\perp}$$

$$\vec a_\perp = |\vec a_\perp| \; \hat a_\perp \qquad \sim \text{vom Betrag } 1 \quad (\text{nach V3})$$

$$\rightsquigarrow \quad |\hat b \times \vec a| \overset{!}{=} |\vec a_\perp|$$

zu 4): $|\vec a \times \vec b| = |\vec a|\,|\vec b|\,|\sin\vartheta|$ ✓

$$|\hat b \times \vec a|^2 = |\vec a_\perp|^2 \overset{\text{S.d.P.}}{=} |\vec a|^2 - |\vec a_\parallel|^2$$

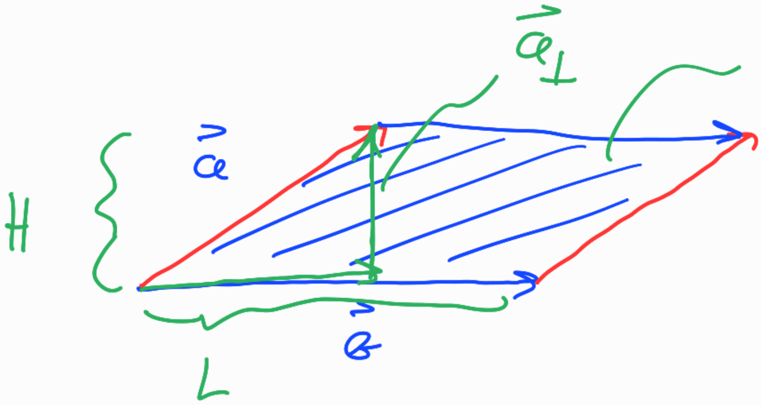
$$= |\vec a|^2 - |\underbrace{\langle \hat b, \vec a\rangle}_{=|\vec a|\langle \hat b, \hat a\rangle = |\vec a|\cos\vartheta}|^2$$

$$= |\vec a|^2 - |\vec a|\cos^2\vartheta$$

$$= |\vec a|^2(1-\cos^2\vartheta) = |\vec a|^2\sin^2\vartheta$$

$$\xrightarrow{\cdot|\vec b|^2} \quad |\vec b \times \vec a|^2 = |\vec a|^2\,|\vec b|^2\sin^2\vartheta$$

zu 5) Flächeninhalt eines Parall.:



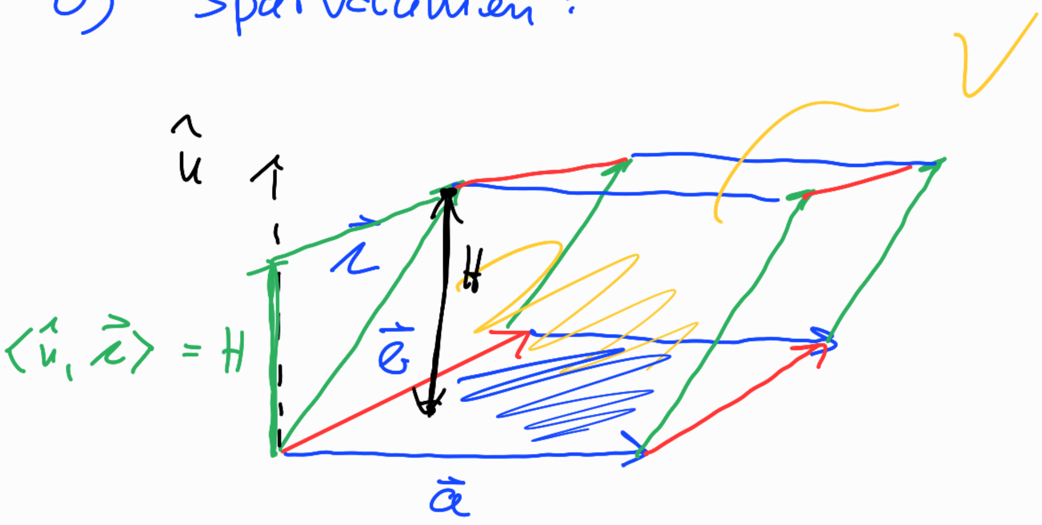
$A = L \cdot H$

$L = |\vec b|$

$H = |\vec a_\perp| = |\hat b \times \vec a|$

$$\rightarrow \quad A = |\vec b|\,|\hat b \times \vec a| = |\vec b \times \vec a|$$

6) Spatvolumen:



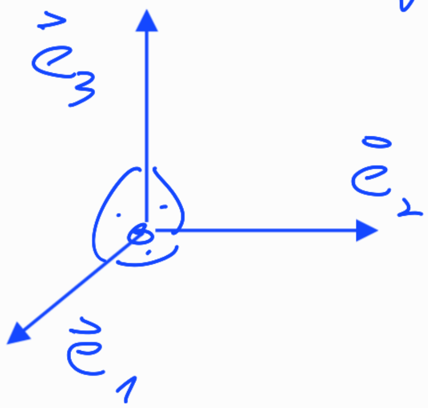
$$V = A \cdot H \quad ; \quad A = |\vec{a} \times \vec{b}|$$

$$H = \langle \hat{u}, \vec{c} \rangle$$

$$\hat{u} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

$$V = \underbrace{|\vec{a} \times \vec{b}|}_{A} \cdot \underbrace{\left|\left\langle \overbrace{\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}}^{\vec{u}}, \vec{c} \right\rangle\right|}_{H} = |\langle \vec{a} \times \vec{b}, \vec{c} \rangle|$$

7) Berechnung in Komp. bzgl. ONB $B=(\vec{e}_1,\vec{e}_2,\vec{e}_3)$



$V3$:
$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$$
$$\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$$
$$\vec{e}_3 \times \vec{e}_1 = \vec{e}_2$$

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_B = \sum_{i=1}^{3} a_i \vec{e}_i$$

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_B = \sum_{j=1}^{3} b_j \vec{e}_j$$

$$\vec{a} \times \vec{b} = \left(\sum_i a_i \vec{e}_i\right) \times \left(\sum_j b_j \vec{e}_j\right)$$

$$= \sum_i \sum_j a_i b_j \, \vec{e}_i \times \vec{e}_j \;\ldots = \sum_i \sum_{j \neq i} a_i b_j$$

$$= \sum_i \sum_{i<j} \left(a_i b_j \, \vec{e}_i \times \vec{e}_j + a_j b_i \underbrace{\vec{e}_j \times \vec{e}_i}_{\ominus \vec{e}_i \times \vec{e}_j}\right)$$

$$= \sum_i \sum_{i<j} (a_i b_j - a_j b_i) \, \vec{e}_i \times \vec{e}_j \quad !$$

$$\underline{\underline{\vec{a} \times \vec{b}}} = (a_1 b_2 - a_2 b_1)\, \overbrace{\vec{e}_1 \times \vec{e}_2}^{= \underline{\underline{\vec{e}_3}}}$$

$$+ (a_1 b_3 - a_3 b_1)\, \underbrace{\vec{e}_1 \times \vec{e}_3}_{= \underline{\underline{-\vec{e}_2}}}$$

$$+ (a_2 b_3 - a_3 b_2)\, \underbrace{\vec{e}_2 \times \vec{e}_3}_{= \underline{\underline{\vec{e}_1}}}$$

$$= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}_B \qquad !$$