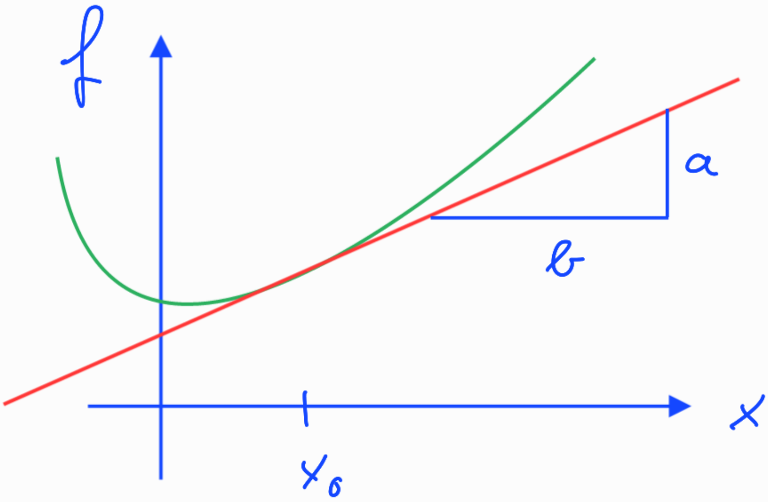
letzte Vrlsg.:

$$\boxed{\begin{aligned}&\text{Ableitung } f'(x_0) \text{ von } f \text{ in } x_0\\ &= \text{Steigung der Tangente an Fktsgraphen in } x_0\end{aligned}}$$



$$\boxed{f'(x_0) = \frac{a}{b} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

Ableitung von $f: D \to \mathbb{R}$

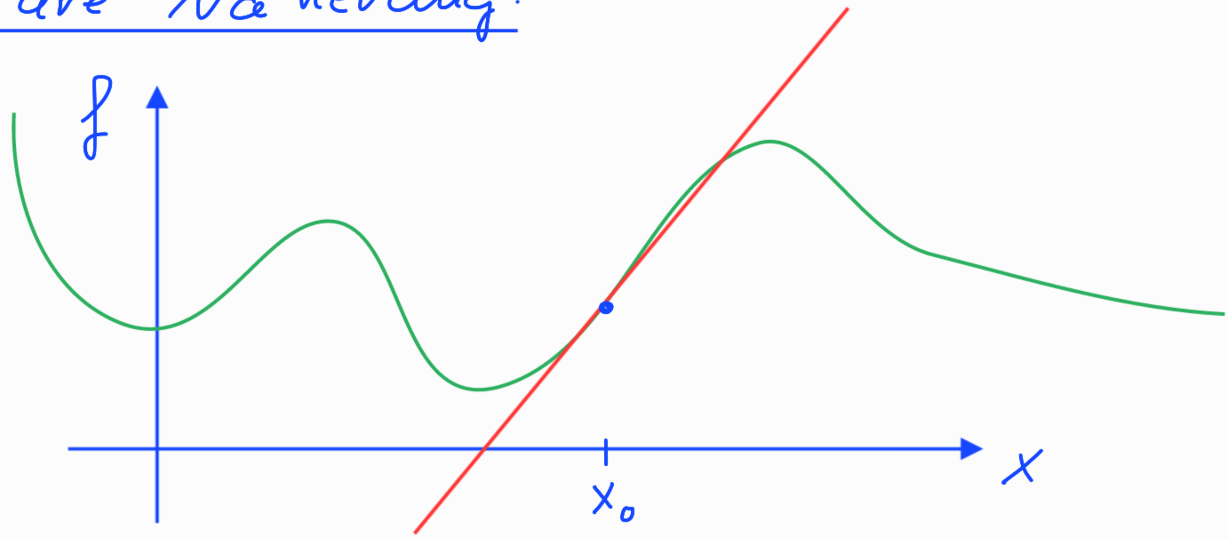
$$= \text{Fkt. } f': D \to \mathbb{R}$$
$$x \mapsto f'(x)$$

alt. Notation:

$$f' = \frac{df}{dx}, \quad f'(x_0) = \frac{df}{dx}(x_0), \quad f'(t) = \dot{f}(t)$$

t Zeit

Lineare Näherung:



$$\boxed{f(x_0+\Delta x) \simeq f(x_0) + f'(x_0)\,\Delta x}$$

element. Berechnung:

$$\underline{(x^2)'} = \frac{1}{h}\left\{ (x+h)^2 - x^2 \right\}$$

$$= \frac{1}{\cancel{h}}\left\{ \cancel{x^2} + 2\cancel{h}x + h^{\cancel{2}} \cancel{-x^2} \right\}$$

$$= 2x + h \xrightarrow{h=0} \underline{2x}$$

Funktionen x^n, x^α, $\sin(x)$, $\cos(x)$, $\exp(x)$, $\ln(x)$, ...

und deren Ableitungen, Ableitungsregeln

• $n \in \mathbb{N}$: $$(x^n)' = \frac{1}{h}\Big\{\underbrace{(x+h)^n}_{(x+h)(x+h)\cdots(x+h)} - x^n\Big\}$$

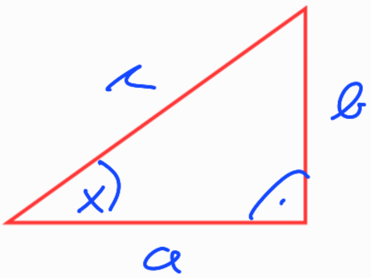
← n mal

$$= \frac{1}{h}\Big\{\cancel{x^n} + n h x^{n-1} + \underbrace{h^2 x^{n-2} + \cdots}_{\mathcal{O}(h^2)} - \cancel{x^n}\Big\}$$

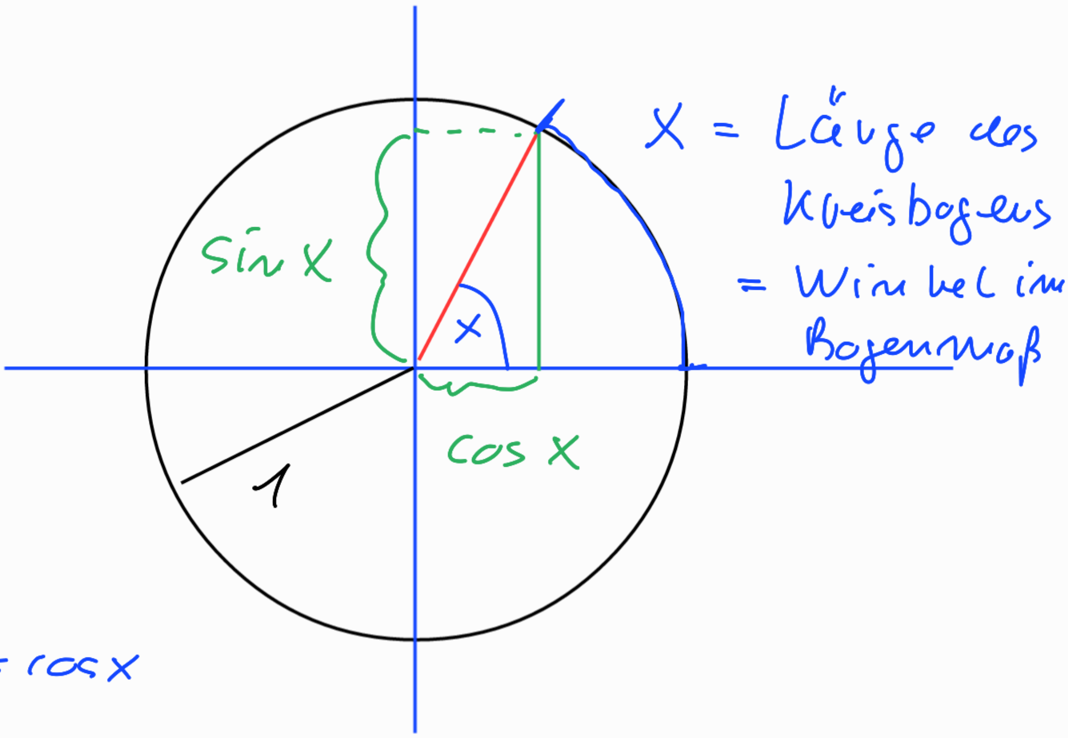
$$= n x^{n-1} + \mathcal{O}(h) \xrightarrow{h=0} n x^{n-1}.$$

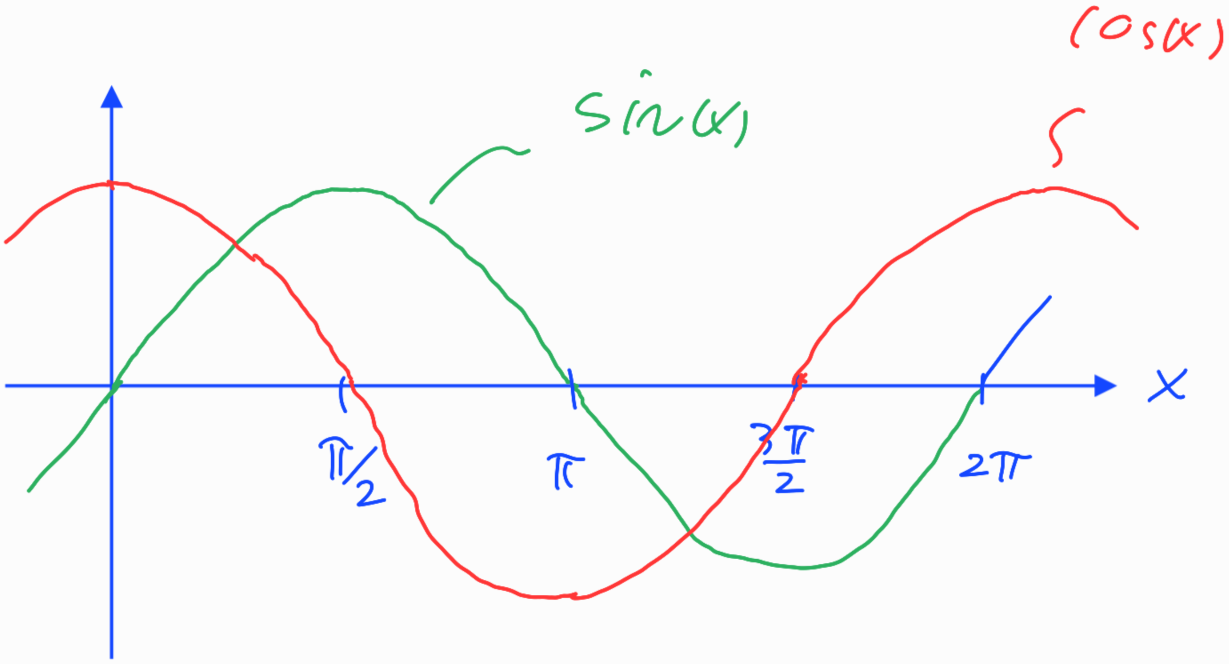
• Trigonometrische $\sin(x)$, $\cos(x)$

geom. Def.

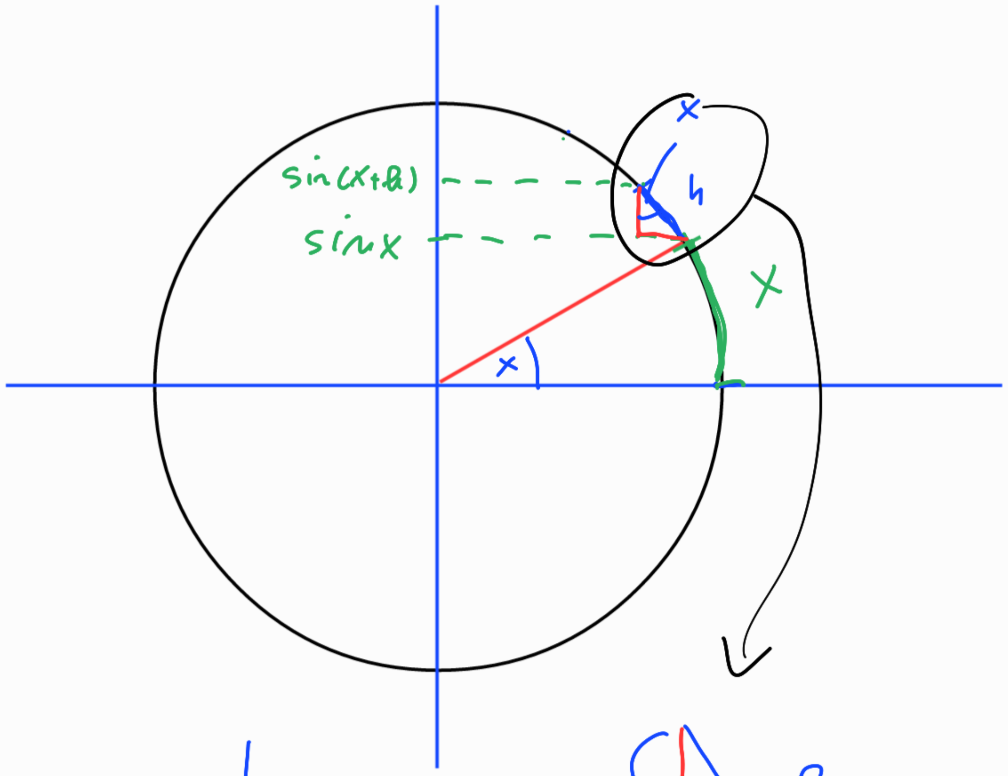


$\frac{b}{c} = \sin x$, $\frac{a}{c} = \cos x$

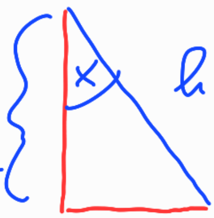




$$(\sin(x))' = \cos(x)$$



$$h \cos x \overset{!}{=} \underbrace{\sin(x+h) - \sin x}$$



$$(\sin x)' = \frac{1}{h}\left(\dots \right) = \frac{1}{\cancel{h}} \cancel{h} \cos x = \cos x \; !$$

$\rightarrow$ $$\boxed{\begin{aligned} (\sin x)' &= \cos x \\ \text{analog: } (\cos x)' &= -\sin x \end{aligned}}$$

- Exponentialfunktion

Was ist die Ableitung von

$$f_a(x) := a^x \qquad ?$$

a ← Basis $a \in \mathbb{R}_+$, x ← Exponent

$$(a^x)' = \frac{1}{h}\left(\underbrace{a^{x+h}}_{= a^x a^h} - a^x\right)$$

$$= \underbrace{\frac{1}{h}(a^h - 1)}_{\text{unabhängig von } x! \ (h \to 0)} a^x$$

$$(a^x)' = c_a \, a^x$$

mit $c_a = \lim_{h \to 0} \frac{1}{h}(a^h - 1)$

Euler: wähle Basis a_0 so, dass

$$c_{a_0} \stackrel{!}{=} 1 \quad !$$

$$\rightarrow \quad (a_0^x)' = a_0^x \quad !$$

→ Euler-Zahl e: bestimmt

durch $$c_e \stackrel{!}{=} 1$$

$$\begin{array}{c} \parallel \\ \lim\limits_{h\to 0} \frac{1}{h}\left(e^{h} - 1\right) \stackrel{!}{=} 1 \end{array}$$

setze $h = \frac{1}{n}$ und betrachte $\overset{\mathbb{N}}{\overset{\in}{n}} \to \infty$

d.h. e bestimmt:

$$\lim_{n\to\infty} n\left(e^{1/n} - 1\right) \stackrel{!}{=} 1$$

betrachte $n \gg 1$:

d.h. für $m \gg 1$:

$$m\left(e^{1/m} - 1\right) \simeq 1$$

$$\hookrightarrow \quad \boxed{e = \lim_{m\to\infty}\left(1 + \tfrac{1}{m}\right)^m}$$

$e = ?$: $\quad m \gg 1$

m mal

$$\left(1 + \tfrac{1}{m}\right)^m = \left(1 + \tfrac{1}{m}\right)\left(1 + \tfrac{1}{m}\right)\cdots\left(1 + \tfrac{1}{m}\right)$$

$$= 1 + \cancel{m}\frac{1}{\cancel{m}} + \underbrace{\frac{\cancel{m}(\cancel{m-1})}{2}}_{\binom{m}{2}}\frac{1}{\cancel{m^2}} + \underbrace{\frac{\cancel{m}(\cancel{m-1})(\cancel{m-2})}{1\cdot 2\cdot 3}}_{\binom{m}{3}}\frac{1}{\cancel{m^3}}$$

$m \to \infty$

$$+ \frac{\cancel{m}(\cancel{m-1})\cdots(\cancel{m-3})}{1\cdot 2\cdot 3\cdot 4}\frac{1}{\cancel{m^4}} + \cdots\cdots$$

$$\boxed{\begin{aligned} e &= 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots \\ &= 2{,}71828\ldots \end{aligned}}$$

Def.: Fakultät: $\ell! := 1 \cdot 2 \cdot 3 \cdot \ldots (\ell-1)\ell$

$0! := 1$

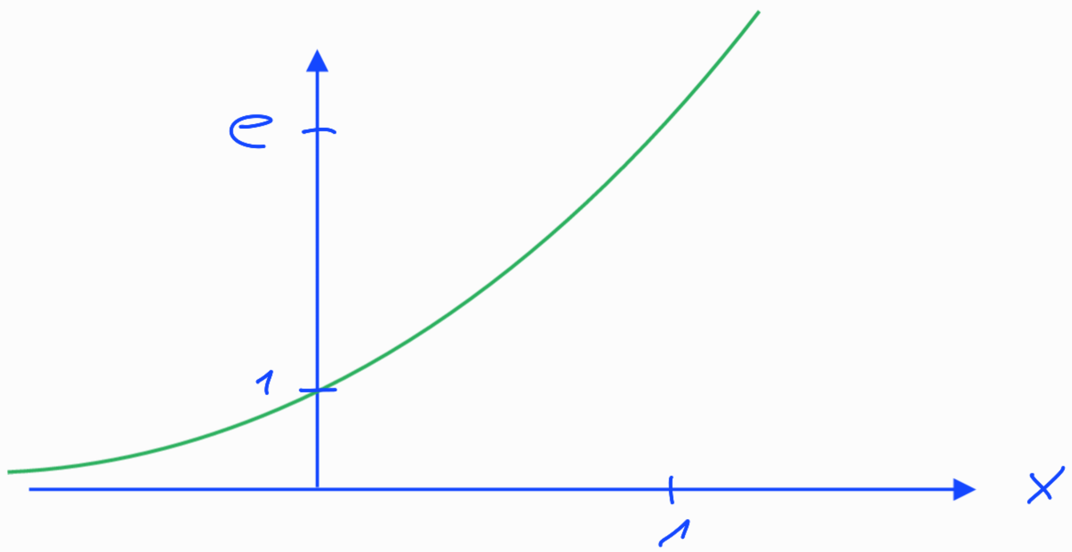
$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

$$\hookrightarrow \boxed{e = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}}$$

<u>Exponentialfunktion exp</u>

$$\boxed{\begin{aligned} \exp : \mathbb{R} &\longrightarrow \mathbb{R}_+ \\ x &\longmapsto \exp(x) := e^x \end{aligned}}$$

$$(\exp)' \overset{!}{=} \exp \qquad (e^x)' \overset{!}{=} e^x$$



Ableitungsregeln:

$f(x) = e^{\lambda x} (\sin(x^2))^5$; $f'(x) = ?$

1) $(f+g)' = f' + g'$

$(\lambda f)' = \lambda f'$ (Linearität)

2) $(f \cdot g)' = f'g + fg'$ (Produktregel)

3) $\left(f/g\right)' = \dfrac{f'g - fg'}{g^2}$ (Quotientenregel)

4) $(f \circ g)' = (f' \circ g) \cdot g'$

$(f(g(x)))' = \underbrace{f'(g(x))}_{\text{äußere}} \underbrace{g'(x)}_{\text{innere}}$ (Kettenregel)

äußere innere Ableitung

5) $\left(f^{-1}(y)\right)' = \dfrac{1}{f'(f^{-1}(y))}$

(Abl. der Umkehrfkt.)

$$f(x) = e^{\lambda x} \cdot \left(\sin(x^2) \right)^5$$

$$f'(x) \overset{3)}{=} \left(e^{\lambda x}\right)' \left(\sin(x^2)\right)^5 + e^{\lambda x} \left(\left(\sin(x^2)\right)^5 \right)'$$

$$= \lambda e^{\lambda x} \left(\sin(x^2)\right)^5 + e^{\lambda x} \, 5 \left(\sin(x^2)\right)^4 \cdot \left(\sin(x^2)\right)'$$

$$= e^{\lambda x} \sin(x^2)^4 \left\{ \lambda \sin(x^2) + 5 \cos(x^2) \, 2x \right\}$$

Warum gelten die Ableitungsregeln?

<u>Linearität</u>

$$\frac{1}{h}\left((f+g)(x+h) - (f+g)(x) \right)$$

$$= \underbrace{\frac{1}{h}\left(f(x+h) - f(x) \right)}_{f'(x)} + \underbrace{\frac{1}{h}\left(g(x+h) - g(x) \right)}_{g'(x)} \quad \checkmark$$

Productregel

$$(fg)'(x) = \frac{1}{h}\Big(\overbrace{f(x+h)}^{(f(x)+f'(x)h)}\,\underbrace{g(x+h)}_{(g(x)+g'(x)h)} - f(x)g(x)\Big)$$

$$= \frac{1}{h}\Big\{\cancel{f(x)g(x)} + \big(f'(x)g(x) + f(x)g'(x)\big)h + \mathcal{O}(h^2) - \cancel{f(x)g(x)}\Big\}$$

$$= f'(x)g(x) + f(x)g'(x) + \cancel{\mathcal{O}(h)}$$

kettingregel:

$$\big(f(g(x))\big)' = \frac{1}{h}\Big(f(\overbrace{g(x+h)}^{g(x)+g'(x)h}) - f(g(x))\Big)$$

$$= \cancel{\frac{1}{h}}\Big\{\underbrace{f(g(x)+g'(x)h)}_{\cancel{f(g(x))} + f'(g(x))\cdot g'(x)\cancel{h}} - \cancel{f(g(x))}\Big\}$$

$$= f'(g(x))\,g'(x)$$

<u>Quotientenregel</u> : $(f/g)' = (f'g - fg')/g^2$

$$\left(f \cdot \frac{1}{g}\right)' = f' \frac{1}{g} + f \underbrace{\left(\frac{1}{g}\right)'}_{-\frac{1}{g^2} \cdot g'} = (f'g - fg')/g^2 \checkmark$$

$$(f \circ f^{-1})(y) \overset{!}{=} y \qquad \Big| \frac{d}{dy}$$

$$f'(f^{-1}(y)) \cdot (f^{-1})'(y) \overset{!}{=} 1 \qquad \Big| \checkmark$$

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} .$$