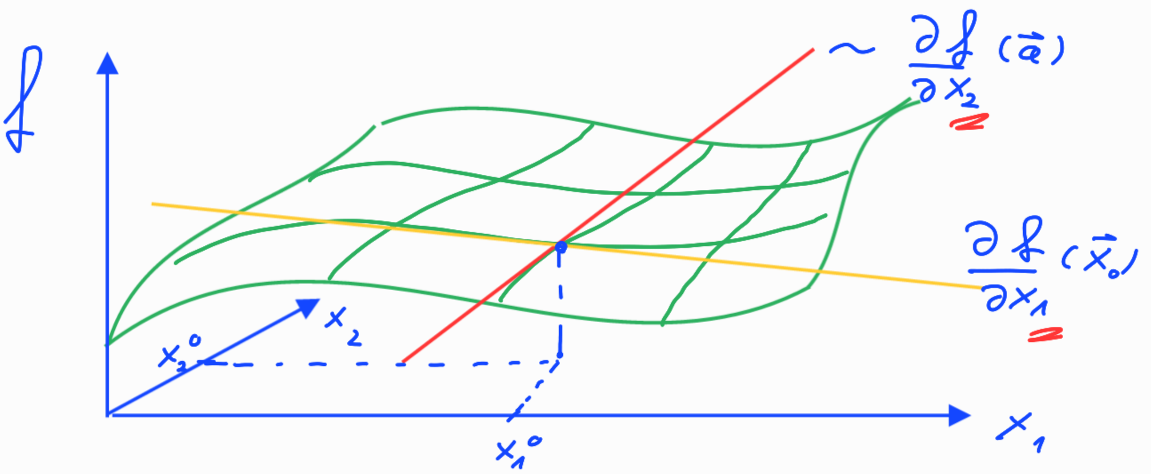
Wdhlg: <u>partielle Ableitung</u>

$$f : D \longrightarrow \mathbb{R} \qquad D \subset \mathbb{R}^n$$
$$\vec{x} \longmapsto f(\vec{x}) = f(x_1, x_2, \dots x_n)$$

$n = 2$



$$\frac{\partial f}{\partial x_e}(\vec{x}) = \Big(f(\underbrace{x_1, \dots, x_{e-1}}_{\text{fest}}, \underset{\uparrow \atop \text{variabel}}{x_e}, \underbrace{x_{e+1}, \dots x_n}_{\text{fest}}) \Big)'$$

$$= \lim_{h \to 0} \frac{1}{h} \Big(f(x_1, \dots, x_e + h, \dots x_n) - f(\vec{x}) \Big)$$

$$= \lim_{h \to 0} \frac{1}{h} \Big(f(\vec{x} + h\vec{e}_e) - f(\vec{x}) \Big)$$

Bsp:
$$\frac{\partial}{\partial x_1}(x_1^2 \cos x_2) = 2x_1 \cos x_2$$
$$\frac{\partial}{\partial x_2}(x_1^2 \cos x_2) = -x_1^2 \sin x_2$$

Lineare Näherung einer Fkt $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^n$

$$\vec{x} = (x_1, \dots, x_n) \longmapsto f(\vec{x})$$

$$= f(x_1, \dots, x_n)$$

$\ulcorner$ $n = 1$:

$$f(x+h) = f(x) + f'(x)\, h$$ $\lrcorner$

$$f(\vec{x} + \vec{h}) = f(x_1 + h_1, x_2 + h_2, \dots, x_n + h_n)$$

$$= f(x_1, x_2 + h_2, \dots, x_n + h_n) + \frac{\partial f}{\partial x_1}(\vec{x})\, h_1$$

$$= f(x_1, x_2, x_3 + h_3, \dots, x_n + h_n) + \frac{\partial f}{\partial x_1}(\vec{x})\, h_1 + \frac{\partial f}{\partial x_2}(\vec{x})\, h_2$$

$$\boxed{f(\vec{x} + \vec{h}) = f(\vec{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x})\, h_i}$$

Gradient!

Def: Gradient von f in $\vec{x}$:

$$\boxed{\operatorname{grad} f(\vec{x}) := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x})\, \vec{e}_i}$$

$B = (\vec{e}_1, \dots, \vec{e}_n)$

$$\operatorname{grad} f(\vec{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\vec{x}) \\ \frac{\partial f}{\partial x_2}(\vec{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\vec{x}) \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}}_{\vec{\nabla} \;:\; \text{"Nabla"}} f(\vec{x})$$

$$\operatorname{grad} f(\vec{x}) = \vec{\nabla} f(\vec{x})$$

$\rightarrow$ Lineare Näherung:

$$\boxed{f(\vec{x} + \vec{h}) = f(\vec{x}) + \langle \operatorname{grad} f(\vec{x}), \vec{h} \rangle}$$

Bsp.: $f(x_1, x_2, x_3) = (x_1 + 1)^2 (x_2 + 1) \cos x_3$

$$\Leftrightarrow \operatorname{grad} f(\vec{x}) = \begin{pmatrix} 2(x_1+1)(x_2+1)\cos x_3 \\ (x_1+1)^2 \cos x_3 \\ -(x_1+1)^2(x_2+1)\sin x_3 \end{pmatrix}$$

bei $\vec{x} = \vec{o}$: $\operatorname{grad} f(\vec{o}) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

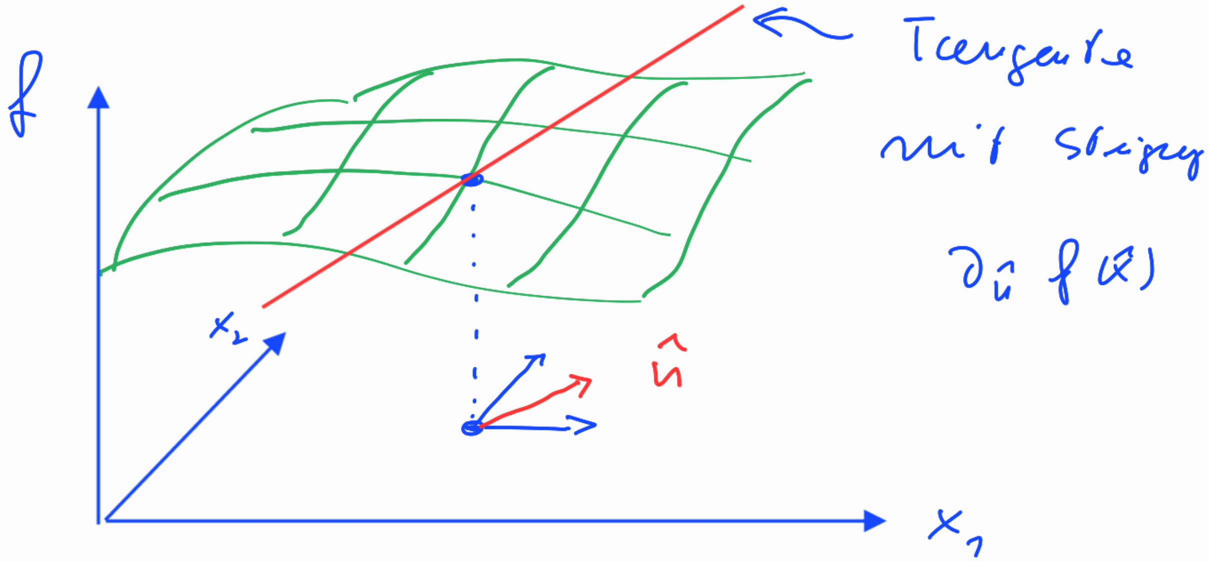
für $|\vec{x}| \ll 1$: $f(\vec{x}) \simeq f(\vec{o}) + \underbrace{\langle \operatorname{grad} f(\vec{o}), \vec{x} \rangle}_{\left\langle \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\rangle}$

$$= 1 + 2x_1 + x_2$$

Richtungsableitung von f in $\vec{x}$ in Richtung $\hat{u}$

$\partial_{\hat{u}} f(\vec{x})$

$(\hat{u} = \mathbb{R}^n , \quad |\hat{u}| = 1)$



$$\boxed{\partial_{\hat{u}} f(\vec{x}) := \lim_{h \to 0} \frac{1}{h} \left(f(\vec{x} + h\hat{u}) - f(\vec{x}) \right)}$$

Berechnung mittels grad $f(\vec{x})$:

$$\lim_{h \to 0} \frac{1}{h} \left(f(\vec{x} + h\hat{u}) - f(\vec{x}) \right) = \cancel{\frac{1}{h}} \left(\cancel{f(\vec{x})} + \langle \operatorname{grad} f(\vec{x}), \cancel{h}\, \hat{u} \rangle - \cancel{f(\vec{x})} \right)$$

$$\boxed{\partial_{\hat{u}} f(\vec{x}) = \langle \operatorname{grad} f(\vec{x}), \hat{u} \rangle}$$

$$\boxed{\partial_{\hat u} f(\vec x) = \langle \operatorname{grad} f(\vec x), \hat u \rangle}$$

$\vec a \to \hat a := \dfrac{\vec a}{|\vec a|}$

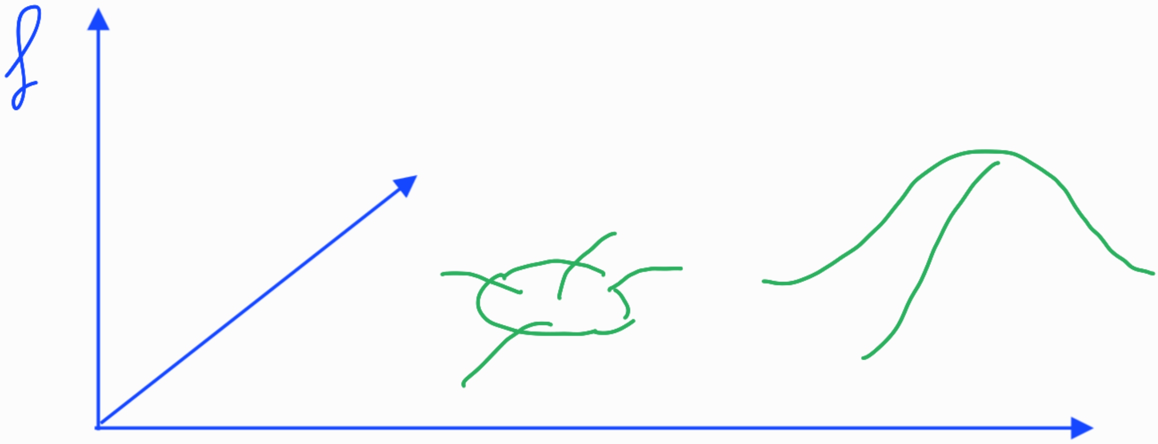
Eigenschaften des Gradienten

1) $\widehat{\operatorname{grad} f(x)}$ = Richtung $\hat n_0$ maximaler Steigung

2) $|\operatorname{grad} f(\vec x)|$ = Steigung in Richtung $\hat n_0$

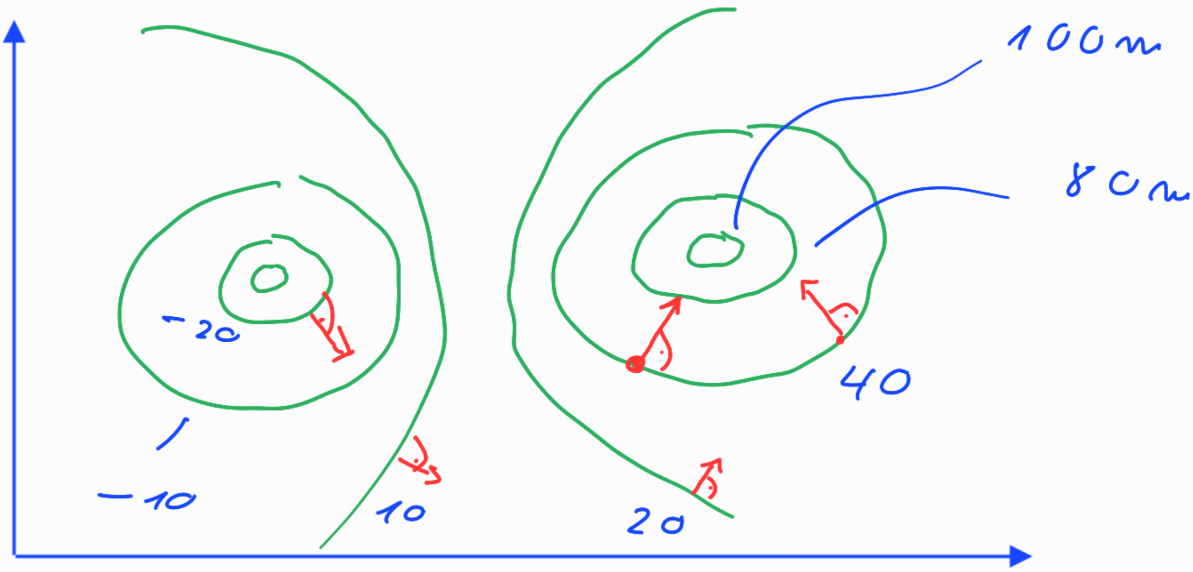
3) $\operatorname{grad} f(\vec x) \perp$ Hyperfläche $f = c$

Bsp.: $f(x_1, x_2)$ = "Höhe über 0"

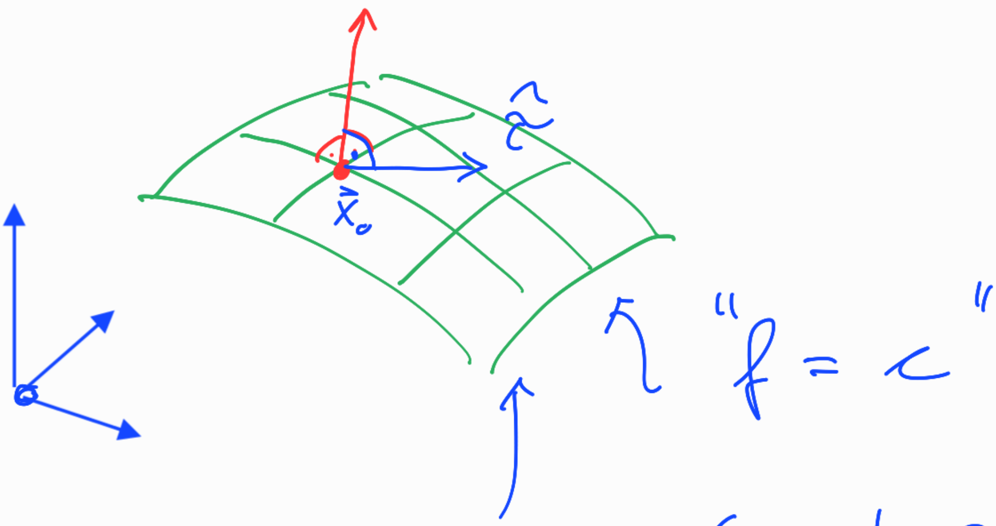


mittels Höhenlinien:

$n=2$



$n=3$



Fläche $F = \{ \vec{x} \mid f(\vec{x}) = c \}$

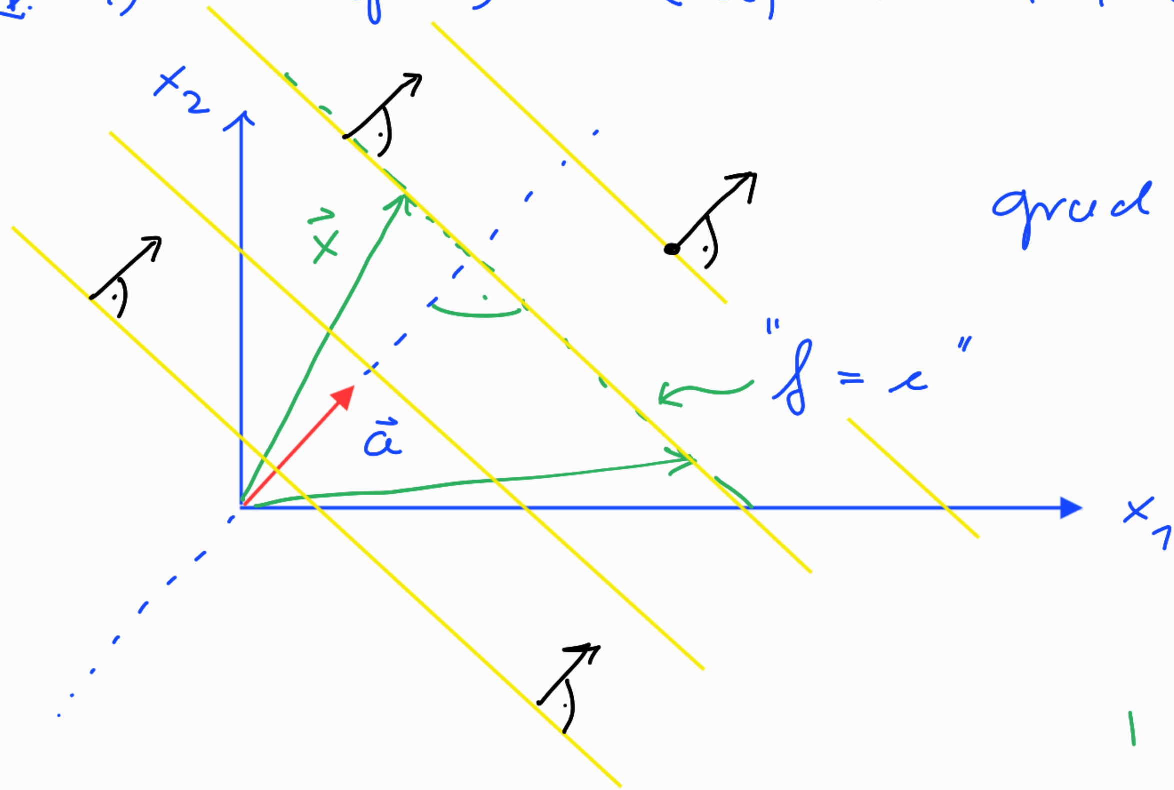
$\hat{\tau}$ tangential zur F in $\vec{x}_0 \in F$

$\Leftrightarrow \quad \partial_{\hat{\tau}} f(\vec{x}_0) \overset{!}{=} 0$

$\Rightarrow \quad \langle \operatorname{grad} f(\vec{x}_0), \hat{\tau} \rangle = 0$

$\Rightarrow \quad \operatorname{grad} f(\vec{x}_0) \perp \hat{\tau} \qquad \longrightarrow 3)$

Bsp.: 1) $f(\vec{x}) = \langle \vec{a}, \vec{x} \rangle = |\vec{a}| \langle \hat{a}, \vec{x} \rangle$



$\operatorname{grad} f(\vec{x}) \overset{!}{=} \vec{a}$

Rechnung 1:

$$f(\vec{x}+\vec{h}) = \langle \vec{a}, \vec{x}+\vec{h} \rangle = \overset{f(\vec{x})}{\overset{\shortparallel}{\langle \vec{a}, \vec{x} \rangle}} + \overset{\langle \operatorname{grad} f(\vec{x}), \vec{h} \rangle}{\overset{\shortparallel}{\langle \vec{a}, \vec{h} \rangle}}$$

$$= \underline{f(\vec{x})} + \langle \underline{\operatorname{grad} f(\vec{x})}, \underline{\vec{h}} \rangle$$

$$\longrightarrow \quad \operatorname{grad} f(\vec{x}) = \vec{a} \quad !$$

Rechnung 2:

$$\operatorname{grad} f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\vec{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\vec{x}) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \vec{a}$$

;

$$\frac{\partial}{\partial x_\ell} \langle \vec{a}, \vec{x} \rangle = \frac{\partial}{\partial x_\ell} \sum_{i=1}^{n} a_i x_i$$

$$= \sum_{i=1}^{n} a_i \frac{\partial x_i}{\partial x_\ell}$$

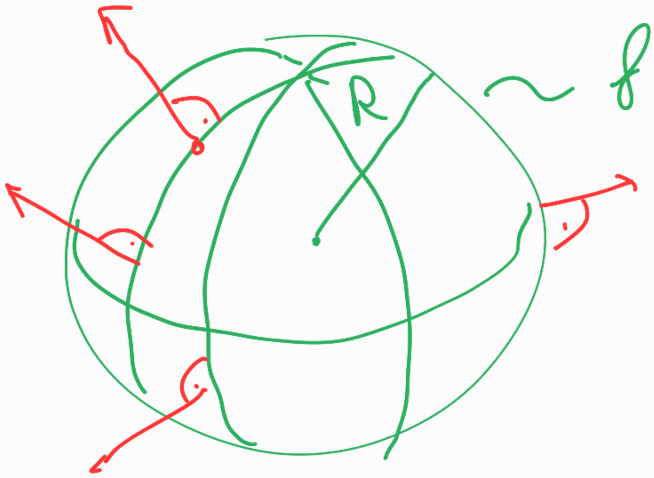
$$\frac{\partial x_i}{\partial x_\ell} = \delta_{i\ell}$$

$$= \sum_{i=1}^{n} a_i \delta_{i\ell}$$

$$= a_\ell \quad \checkmark$$

2) $n = 3$, $\vec{x} = \vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$;

$f(\vec{r}) = |\vec{r}|$; $\text{grad}\, |\vec{r}| = ?$



$\text{grad}\, |\vec{r}| \overset{!}{=} \hat{r}$

Rechnung:

$$\frac{\partial}{\partial x_1} |\vec{r}| = \frac{\partial}{\partial x_1} \sqrt{x_1^2 + x_2^2 + x_3^2} = \frac{\cancel{2} x_1}{\cancel{2} \sqrt{x_1^2 + \dots}}$$

$$= \frac{x_1}{|\vec{r}|} \quad ;$$

$$\rightarrow \text{grad}\, |\vec{r}| = \frac{1}{|\vec{r}|} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\vec{r}} = \frac{\vec{r}}{|\vec{r}|} = \hat{r} \, .$$

Bsp 3) $\quad f(\vec r) \overset{!}{=} g(|\vec r|)$

$$\longrightarrow \quad \boxed{\operatorname{grad} f(\vec r) = g'(|\vec r|) \cdot \hat r}$$

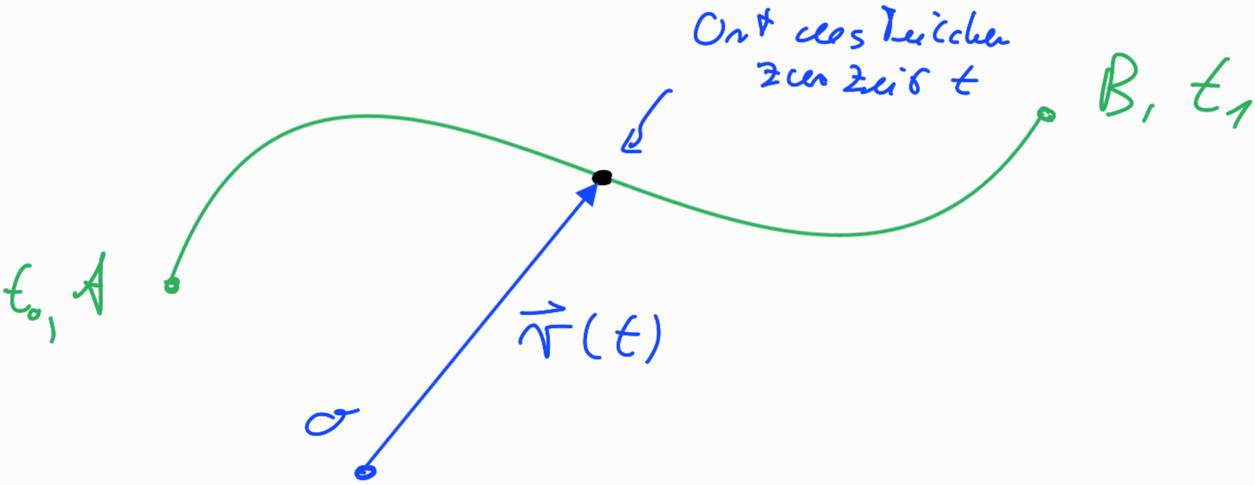
$$\frac{\partial}{\partial x_\ell} f(\vec r) = \frac{\partial}{\partial x_\ell} g(|\vec r|) = g'(|\vec r|) \cdot \underbrace{\frac{\partial |\vec r|}{\partial x_\ell}}_{= \frac{x_\ell}{|\vec r|}}$$

$$\rightarrow \operatorname{grad} f(\vec r) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \partial f / \partial x_3 \end{pmatrix} = \overbrace{\frac{1}{|\vec r|} \cdot \vec r}^{\hat r} \cdot g'(|\vec r|)$$

<u>Kinematik</u>: Bahn, Geschwindigkeit, Beschleunigung eines Massenpunkts im Raum

Ortsvektor in kart. Koord.

$$\vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



$$\boxed{\underline{\text{Bahn}} = \text{Abb. } \vec{r} : [t_0, t_1] \longrightarrow \mathbb{R}^3, \quad t \longmapsto \vec{r}(t)}$$