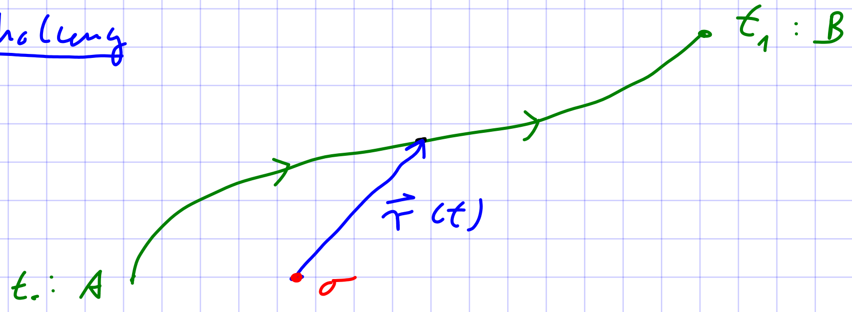
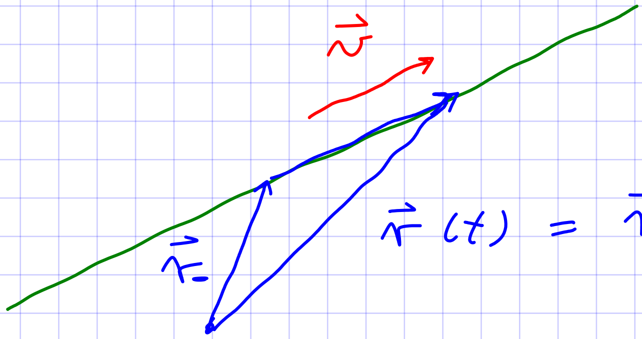
Wiederholung



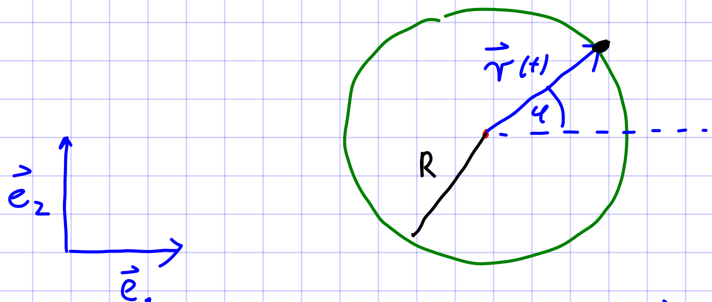
$$\text{Bahn} \;\hat{=}\; \text{Abb. } \vec{r} : [t_0, t_1] \to \mathbb{R}^3$$
$$t \mapsto \vec{r}(t)$$

Beispiele: 1) geradlinig gleichförmige Bewegung:



$$\vec{r}(t) = \vec{r}_0 + \vec{v}\,t$$

2) gleichförmige Kreisbewegung



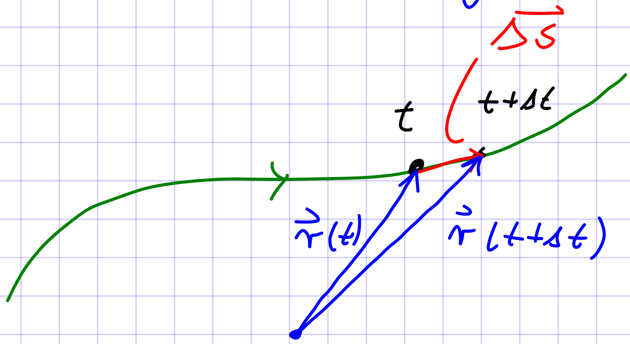
$$\varphi(t) = \omega t$$

$$\rightarrow \vec{r}(t) = R \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}$$

Umlaufzeit: $T = \frac{2\pi}{\omega} \longrightarrow \vec{r}(t+T) = \vec{r}(t)$

momentane Geschwindigkeit

$$\vec{v}(t) = \frac{\text{Ortsänderung}}{\Delta t} = \frac{\vec{\Delta s}}{\Delta t}$$



$$\Delta t \to 0$$

Vektoren !

$$\to \quad \vec{v}(t) = \frac{\vec{\Delta s}}{\Delta t} = \frac{1}{\Delta t}\left(\vec{r}(t+\Delta t) - \vec{r}(t)\right)$$

Differenzenquot.

Skalarmultiplikation

$$\to \quad \vec{v}(t) = \dot{\vec{r}}(t) = \frac{d}{dt}\vec{r}(t) = \lim_{h\to 0}\frac{1}{h}\left(\vec{r}(t+h) - \vec{r}(t)\right)$$

$$\to \quad \boxed{\vec{v} = \dot{\vec{r}} = \frac{d}{dt}\vec{r}}$$

Beispiele: 1) gerad. gl. Bewegung: $\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t$

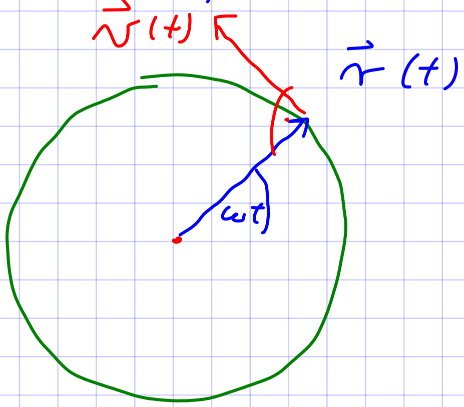
$$\to \quad \vec{v}(t) = \frac{d}{dt}\left(\vec{r}_0 + \vec{v}_0 t\right) = \vec{v}_0$$

konstant!

2) gl. Kreisbewegung: $\vec{r}(t) = R\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix}$

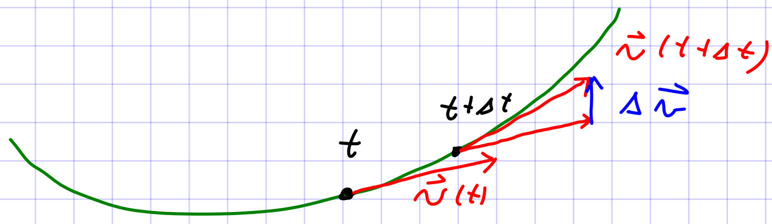
$$\to \quad \underline{\underline{\vec{v}(t)}} = \frac{d}{dt}R\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix} = R\begin{pmatrix}\frac{d}{dt}\cos\omega t\\ \frac{d}{dt}\sin\omega t\end{pmatrix}$$

$$= R\begin{pmatrix}-\omega\sin\omega t\\ \omega\cos\omega t\end{pmatrix} = \underline{\underline{\omega R}}\begin{pmatrix}-\sin\omega t\\ \cos\omega t\end{pmatrix}$$



momentane Beschleunigung

$$\vec a(t) = \frac{\text{Gesch.-änderung}}{\text{Zeit}}$$



$$\vec a(t) = \frac{1}{\Delta t}\left(\vec v(t+\Delta t) - \vec v(t)\right)$$

$$\rightarrow \quad \vec a(t) = \dot{\vec v}(t) = \frac{d}{dt}\vec v(t) = \frac{d^2}{dt^2}\vec r(t)$$

$$\boxed{\vec a = \dot{\vec v} = \ddot{\vec r}}$$

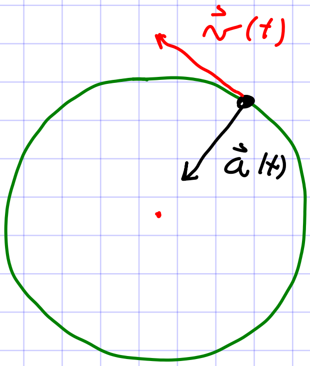
Beispiele:

1) $\vec r(t) = \vec r_0 + \vec v_0 t$

$$\rightarrow \quad \vec v(t) = \vec v_0 \quad \rightarrow \quad \vec a(t) = \vec 0 \quad ✓$$

2) $\vec r(t) = R\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix} \rightarrow \vec v(t) = \omega R\begin{pmatrix}-\sin\omega t\\ \cos\omega t\end{pmatrix}$

$$\rightarrow \quad \vec a(t) = -\omega^2 R\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix}$$



Zentripetalbeschleunigung

$\vec a \parallel \vec r$

3) Bewegung konstanter Beschleunigung: $\vec a(t) \overset{!}{=} \vec g$

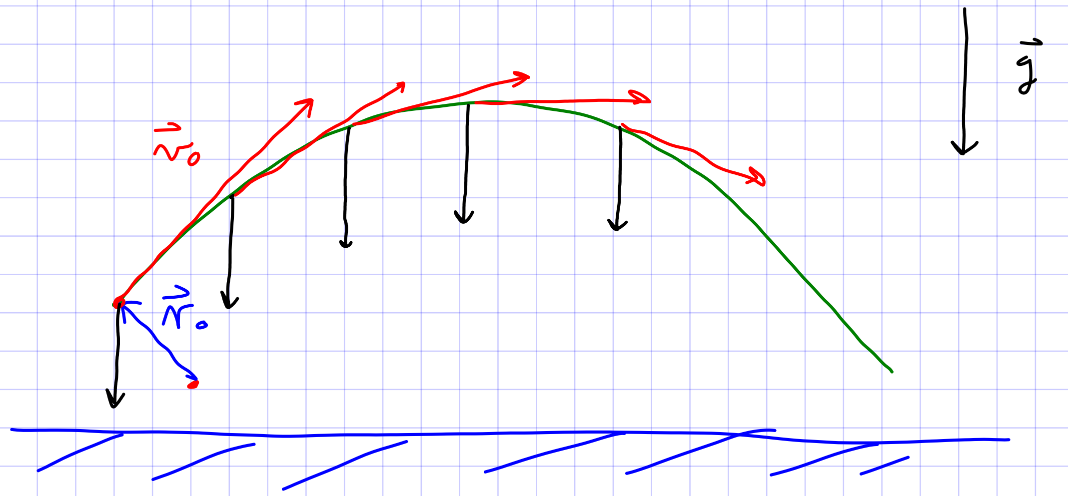
$$\boxed{\vec r(t) = \vec r_0 + \vec v_0 t + \frac{1}{2}\vec g t^2}$$

$$\Rightarrow \quad \vec v(t) = \vec v_0 + \vec g t$$

$$\rightarrow \quad \vec a(t) = \vec g \quad ✓$$

$\Gamma \quad \cancel{m}\vec a(t) = \vec F_g(t) = \cancel{m}\vec g(t)$

$\vec a(t) = \vec g \quad !$



Partielle Ableitung vektorwertiger Funktionen

$$\vec{f} : \mathbb{R}^n \longrightarrow V \quad (= \mathbb{R}^3, \mathbb{R}^m, \dots)$$
$$\vec{x} \longmapsto \vec{f}(\vec{x})$$

$$\longrightarrow \quad \frac{\partial \vec{f}}{\partial x_i}(\vec{x}) = \lim_{h \to 0} \frac{1}{h} \left(\vec{f}(\vec{x} + h\vec{e}_i) - \vec{f}(\vec{x}) \right)$$
$$= \left(\vec{f}(x_1, \dots, x_i, \dots x_n) \right)'$$

Beispiel: Polarkoordinaten:

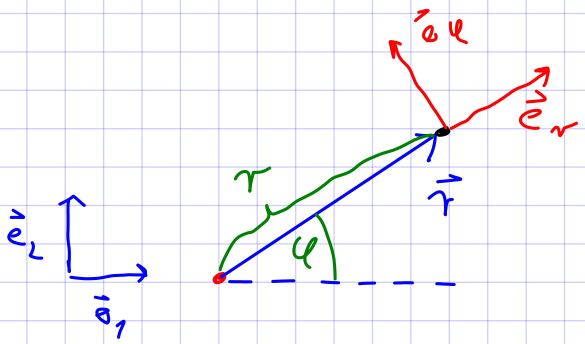


Abb: $\vec{r} : \mathbb{R}_+ \times [0, 2\pi[\longrightarrow \mathbb{R}^2$

$$(r, \varphi) \longmapsto \vec{r}(r,\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$

lokale Basis $\vec{e}_r, \ \vec{e}_\varphi$

$$\longrightarrow \quad \boxed{\vec{e}_r := \frac{1}{\left|\frac{\partial \vec{r}}{\partial r}\right|} \frac{\partial \vec{r}}{\partial r} \quad ; \quad \vec{e}_\varphi := \frac{1}{\left|\frac{\partial \vec{r}}{\partial \varphi}\right|} \frac{\partial \vec{r}}{\partial \varphi}}$$

$\vec{e}_r$: $\frac{\partial \vec{r}}{\partial r} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$; $\vec{e}_\varphi$: $\frac{\partial \vec{r}}{\partial \varphi} = r \begin{pmatrix} -\cos\varphi \\ \sin\varphi \end{pmatrix}$

$1 = \left|\frac{\partial \vec{r}}{\partial r}\right| \leadsto \vec{e}_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$; $\Rightarrow \left|\frac{\partial \vec{r}}{\partial \varphi}\right| = r$

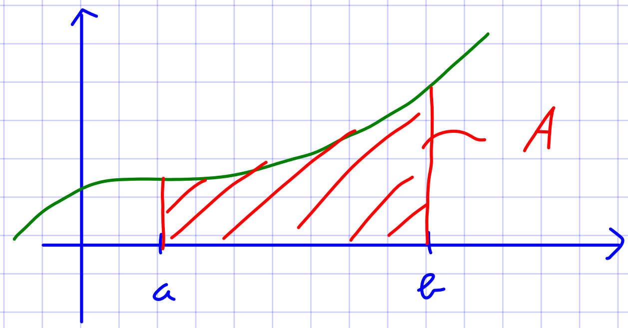
$\Rightarrow \vec{e}_\varphi = \begin{pmatrix} -\cos\varphi \\ \sin\varphi \end{pmatrix}$ ✓

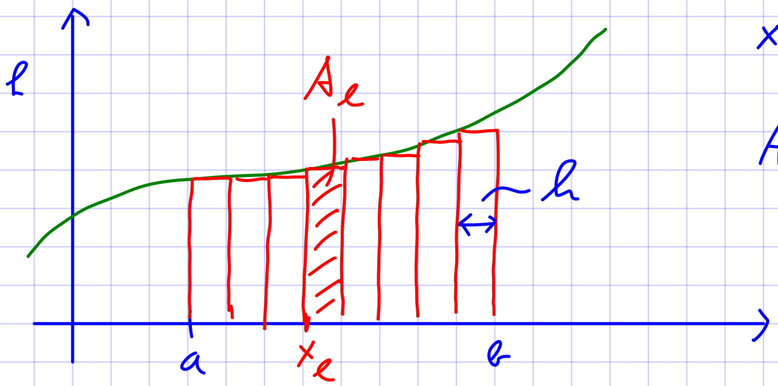
Integral

einer Funktion $f(x)$ in den Grenzen a und b:

$$\int_a^b f(x)\,dx =: A$$

geometrisch:





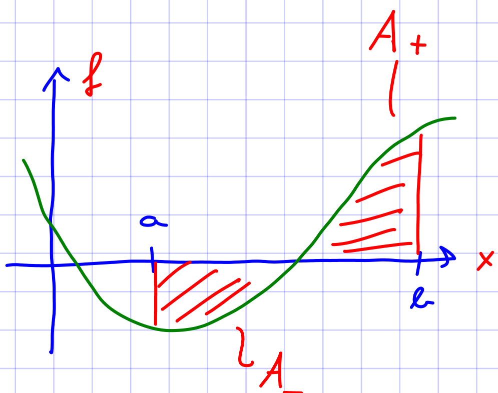
$$x_\ell = a + \ell h$$

$$A_\ell = h\, f(x_\ell) = h\, f(a + \ell h)$$

$a \le b$:

$$\boxed{\int_a^b f(x)\,dx = \lim_{h \to 0} \sum_\ell A_\ell = \lim_{h \to 0} \sum_{\ell=0}^{\frac{b-a}{h}} h\, f(a + \ell h)}$$

$$\int_a^b f(x)\,dx$$



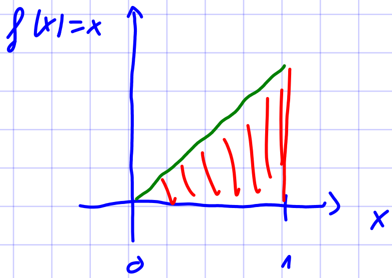
$$\int_a^b f(x) = A_+ - A_-$$

Def: $b < a$:

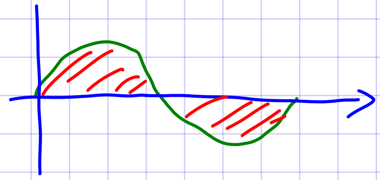
$$\int_a^b f(x)\,dx := -\int_b^a f(x)\,dx$$

Bsps:

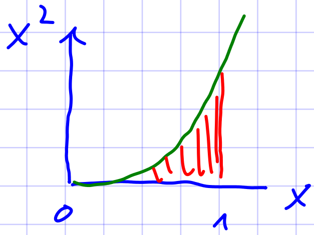
- $$\int_0^1 x\,dx = \frac{1}{2}$$



- $$\int_0^{2\pi} \sin x\,dx = 0$$



- $$\int_0^1 x^2\,dx = 1/3$$



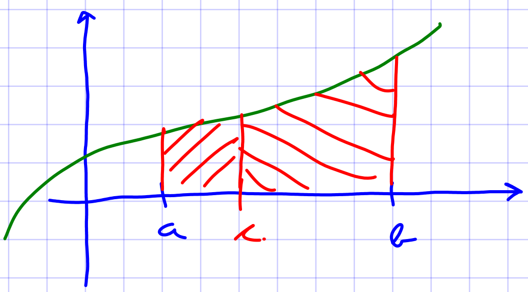
Eigenschaften des Integrals

1) Linearität:

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b (\lambda f)(x)\,dx = \lambda \int_a^b f(x)\,dx$$

2) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



(auch $c < a$, $c > b$!)

Hauptsatz der Differenzial und Integralrechnung

(HDI)

Def: F ist eine Stammfunktion von f
g. d. w. $F' = f$

(i) mit $F(x)$ Stammfkt zu f auch
$$\tilde{F}(x) := F(x) + c$$
Stammfkt zu f ✓

(ii) sind F und $\tilde{F}$ Stammfunktionen zu f, dann gilt
$$F(x) = \tilde{F}(x) + c$$

⌐ zu (ii): $g := \tilde{F} - F$

$$\rightsquigarrow \quad g' = (\tilde{F} - F)' = \tilde{F}' - F' = f - f = 0$$

also $g'(x) = 0 \rightsquigarrow$ d.h. $g(x) = c = \tilde{F}(x) - F(x)$

$$\longrightarrow \quad \tilde{F}(x) = F(x) + c \quad !$$
⌟

HDI

1) $$F_u(x) := \int_u^x f(\tilde{x})\, d\tilde{x}$$ ist Stammfkt. zu f

2) $$\int_a^b f(x)\,dx = F(b) - F(a) \quad \left(= F(x)\Big|_a^b \right)$$

wobei $F(x)$ Stammfkt. zu f