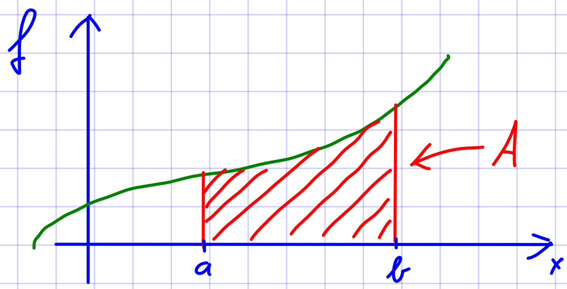
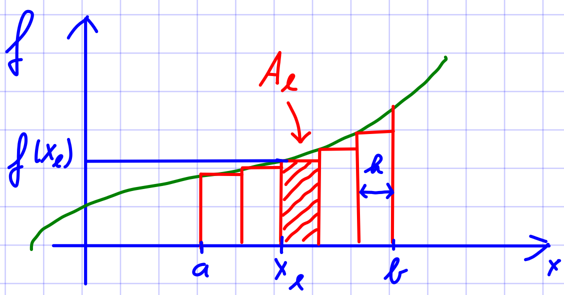
Wiederholung:

Integral: $\displaystyle\int_a^b f(x)\,dx = A = \lim_{h\to 0} \sum_\ell A_\ell = \lim_{h\to 0} \sum_\ell f(x_\ell)\,h$

$a < b$





falls $a > b$: $\displaystyle\int_a^b f(x)\,dx := -\int_b^a f(x)\,dx$

Eigenschaften

- $\displaystyle\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- $\displaystyle\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$
- $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Def.: F Stammfunktion zu f

$:\Longleftrightarrow \quad F' = f$

(i) $F(x)$ Stammfkt. zu f

$\Longrightarrow \quad \tilde{F}(x) := F(x) + c$ Stammfkt. zu f

(ii) F und $\tilde{F}$ Stammfkt.en zu f

$\Longrightarrow \quad \tilde{F}(x) - F(x) = c$ konstant

Hauptsatz der Differenzial- und Integralrechnung (HDI)

1) $F_u(x) := \int_u^x f(\tilde{x})\,d\tilde{x}$ ist eine Stammfunktion von f

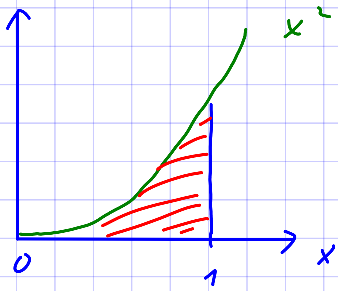
2) $\int_a^b f(x)\,dx = F(b) - F(a) \quad \left(= F(x)\Big|_a^b\right)$

wobei F eine Stammfunktion von f

Beispiele:

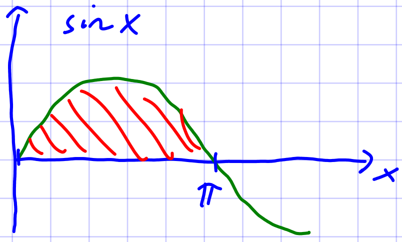
- $\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \underline{\underline{\frac{1}{3}}}$

  $\Leftarrow \left(\frac{x^3}{3}\right)'$: $\frac{x^3}{3}$ ist S.F. zu x^2



- $\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = \underline{\underline{2}}$

  $\Leftarrow (-\cos x)'$: $-\cos x$ ist S.F. zu $\sin x$



- $\int_a^b \sin(kx)\,dx = -\frac{\cos(kx)}{k}\Big|_a^b = \frac{\cos(ka) - \cos(kb)}{k}$

  $\Leftarrow \left(-\frac{\cos(kx)}{k}\right)'$

$$F \underset{\text{integrieren}}{\overset{\text{ableiten}}{\rightleftarrows}} f$$

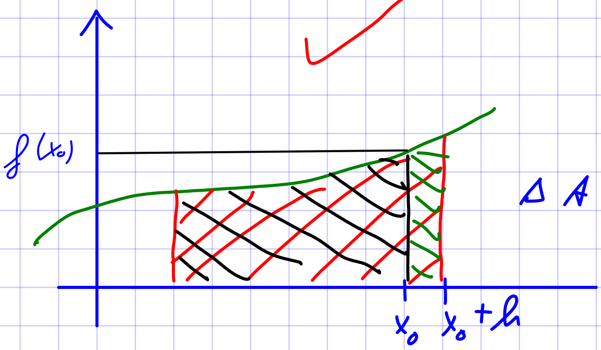
f	$h'(x)$	x^α	$1/x$	e^x	$\sin x$	$\cos x$	$\ln x$	
F	$h(x)$	$\frac{x^{\alpha+1}}{\alpha+1}$	$\ln x$	e^x	$-\cos x$	$\sin x$	$x\ln x - x$	

$\alpha \neq -1$

zum HDI:

1) Bel.: $F_u(x) = \int_u^x f(\tilde{x})\,d\tilde{x}$ ist S.F. zu f!

z.z.: $f(x_0) \overset{!}{=} F_u(x_0)' = \frac{1}{h}\Big(\underbrace{F_u(x_0+h) - F_u(x_0)}_{\Delta A}\Big) = f(x_0)$



$\Delta A = h \cdot f(x_0)$

2) Bel.: $\int_a^b f(x) \overset{!}{=} F(b) - F(a)$

$F - F_u \overset{!}{=} c$, daher $F(b) - F(a) = F_u(b) - F_u(a)$

$$= \int_u^b f(\tilde{x})\,d\tilde{x} - \int_u^a f(\tilde{x})\,d\tilde{x} = \int_a^u f(\tilde{x})\,d\tilde{x} + \int_u^b f(\tilde{x})\,d\tilde{x} = \int_a^b f(x)\,dx$$

Bezeichnungen:

unbestimmtes Integral $\int f(x)\,dx \;\hat{=}\; F(x)$ Stammfkt. zu f

"$\int x^2\,dx = \frac{x^3}{3}$"

$F' = f$

Substitutionsregel:

$$\boxed{\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(y))\,g'(y)\,dy}$$

$x = g(y)$

umkehrbare Funkt. $g: [a, e] \to \mathbb{R}$

Beispiele:

- $$\int_0^1 (3x-1)^4\, dx \overset{\uparrow}{=} \int_{-1}^{2} y^4 \cdot \tfrac{1}{3}\, dy =$$

$$x = \tfrac{1}{3}(y+1) =: g(y)$$

$$\to g'(y) = 1/3$$

$$= \frac{1}{3}\int_{-1}^{2} y^4\, dy = \frac{1}{3}\,\frac{y^5}{5}\Big|_{-1}^{2} = \frac{1}{3\cdot 5}(32+1) = \frac{11}{5} \checkmark$$

⌈ subst. $3x-1 \overset{!}{=} y$

$$\hookrightarrow \frac{dy}{dx} = (3x-1)' = 3 \to dx = \frac{1}{3}\, dy$$

- Strahlungsleistung eines schwarzen Körpers

$$? = u(T) = \alpha_3 \int_0^\infty d\nu\; \underset{\substack{\uparrow \\ QM}}{h\nu}\; \underset{\substack{\uparrow \\ E.D.}}{\nu^2}\; \underbrace{\frac{1}{e^{h\nu/kT}-1}}_{\text{st. Phys.}}$$

$$\underset{\nu = \frac{kT}{h}x}{=} \alpha\cdot \underbrace{\frac{kT}{h}}_{=g'(x)} \int_0^\infty dx\; h\left(\frac{kT}{h}\right)^3 \frac{x^3}{e^x-1}$$

$$\to u(T) = c\, T^4 \cdot \underbrace{\int_0^\infty dx\, \frac{x^3}{e^x-1}}_{\text{unabhängig von } T!\; = c_1} = c_2\, T^4$$

Partielle Integration:

$$\int_a^b f'(x)\,g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)\,g'(x)\,dx$$

[warum?

$$\int_a^b f' g \,dx \overset{!}{=} \int_a^b \{(f g)' - f \cdot g'\}\,dx$$

$$= \int_a^b (fg)'\,dx - \int_a^b f g'\,dx$$

$$= f g\Big|_a^b - \int_a^b f g'\,dx$$

Beispiel:

$$\int_0^\infty x e^{-x}\,dx = \int_0^\infty \underset{=}{\overset{f'}{e^{-x}}}\ \underset{=}{\overset{g}{x}}\,dx \overset{P.I.}{=} \overbrace{-e^{-x} \cdot x}^{fg}\Big|_0^\infty + \int_0^\infty \underset{f}{e^{-x}} \cdot \underset{g'}{1}\,dx$$

$$e^{-x} = (\underbrace{-e^{-x}}_{f})'$$

$$= 0 + \int_0^\infty e^{-x}\,dx = (-e^{-x})\Big|_0^\infty = +1$$

„Ableitungs-Trick":

$$I(n) := \int_0^\infty x^n e^{-x}\,dx\,; \quad n \in \mathbb{N}$$

$$I(1) = \int_0^\infty x e^{-x}\,dx = 1$$

$$I(0) = \int_0^\infty e^{-x}\,dx = 1$$

$$I(2) = ?$$

$n=1$:

$$I(1) = \int_0^\infty \underbrace{x\, e^{-x}}_{\| \; -\frac{d}{d\lambda} e^{-\lambda x}\big|_{\lambda=1}} dx = \int_0^\infty (-)\frac{d}{d\lambda} e^{-\lambda x}\Big|_{\lambda=1} dx$$

$$= (-)\frac{d}{d\lambda}\Bigg(\underbrace{\int_0^\infty e^{-\lambda x} dx}_{(-)\frac{e^{-\lambda x}}{\lambda}}\Bigg)\Bigg|_{\lambda=1} = \frac{d}{d\lambda}\Bigg(\frac{1}{\lambda}\underbrace{\left(e^{-\lambda x}\right)\Big|_0^\infty}_{\underline{\underline{-1}}}\Bigg)_{\lambda=1}$$

$$\to I(1) = \int_0^\infty x\, e^{-x} dx = -\frac{d}{d\lambda}\frac{1}{\lambda}\Big|_{\lambda=1} = 1 \quad \checkmark$$

$$I(n) = \int_0^\infty x^n e^{-x} dx = (-1)^n \frac{d^n}{d\lambda^n}\underbrace{\int_0^\infty e^{-\lambda x} dx}_{+1/\lambda}\Bigg|_{\lambda=1}$$

$$\to I(n) = (-1)^n \frac{d^n}{d\lambda^n}\frac{1}{\lambda}$$

n	0	1	2	3	4	5	6
$I(n)$	1	1	2	$2\cdot 3$	$4!$	$5!$	$6!$

$$+\frac{d^2}{d\lambda^2}\frac{1}{\lambda} = -\frac{d}{d\lambda}\frac{1}{\lambda^2} = +2\frac{1}{\lambda^3} \qquad I(n) = n!$$

$$-\frac{d^3}{d\lambda^3}\frac{1}{\lambda} = 2\cdot 3\ \frac{1}{\lambda^4}\Big|_{\lambda=1}$$

$$+\frac{d^4}{d\lambda^4}\frac{1}{\lambda} = \frac{d}{d\lambda}\frac{1}{\lambda^4}(-2\cdot 3) = 1\cdot 2\cdot 3\cdot 4\ \frac{1}{\lambda^4}$$

$$I(n) = \int_0^\infty x^n e^{-x} \, dx = n! \qquad n \geq 1$$

$$I(0) = \int_0^\infty e^{-x} \, dx = 1$$

→ Def:

$$n! := I(n) \quad \longrightarrow \quad 0! = I(0) = 1$$

Gamma-Fkt:

$$\boxed{\Gamma(n) := \int_0^\infty x^{n-1} e^{-x} \, dx}$$

Euler

$$\Pi(n) := \int_0^\infty x^n e^{-x} \, dx$$

Gauß