Vektoranalysis

bisher: Weg, Wegintegral, Vektorfeld

Weg: $c : [a, b] \longrightarrow \mathbb{R}^3$

$$u \longmapsto \vec{c}(u)$$

Vektorfeld: $\vec{A} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

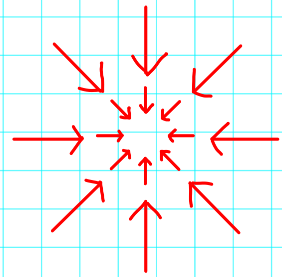
$$\vec{r} \longmapsto \vec{A}(\vec{r})$$

Wegintegral:

$$\int_c \vec{A}\, d\vec{\ell} := \int_a^b \langle \vec{A}(\vec{c}(u)), \vec{c}\,'(u) \rangle \, du$$

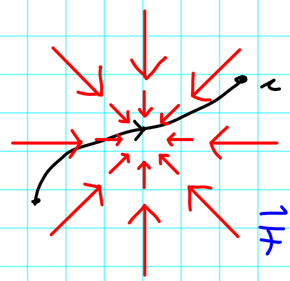
$$\int_c |d\vec{\ell}| := \int_a^b |\vec{c}\,'(u)| \, du$$

Kraftfeld $\vec{F}$



$\rightarrow$ **Arbeit:**

$$W = -\int_c \vec{F}\, d\vec{\ell}$$



heute:

Flächen und Flächenintegrale

Flächenstück ("Fläche") im Raum $\mathbb{R}^3$:

Parametrisierung

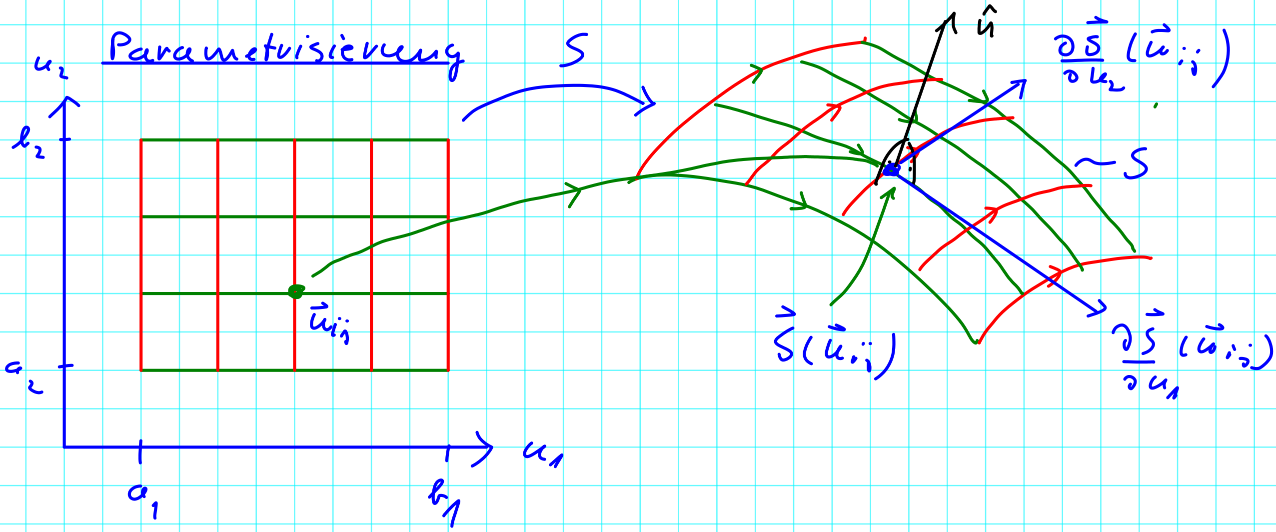


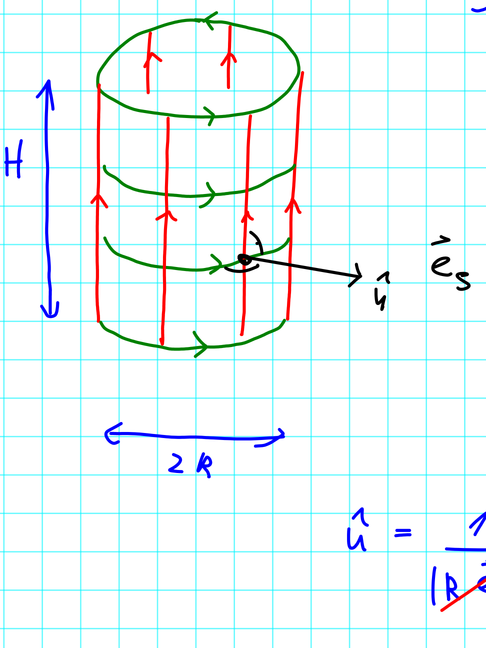
Abb.: $\quad S : [a_1, b_1] \times [a_2, b_2] \longrightarrow \mathbb{R}^3$

$$\vec u = (u_1, u_2) \longmapsto \vec S(\vec u)$$

↪ Normalenvektor:

$$\hat n(\vec u) = \frac{1}{\left| \frac{\partial \vec S}{\partial u_1} \times \frac{\partial \vec S}{\partial u_2} \right|} \frac{\partial \vec S}{\partial u_1}(\vec u) \times \frac{\partial \vec S}{\partial u_2}(\vec u)$$

Beispiele: Zylindermantel:



$$S : [0, 2\pi] \times [0, H] \rightarrow \mathbb{R}^3$$

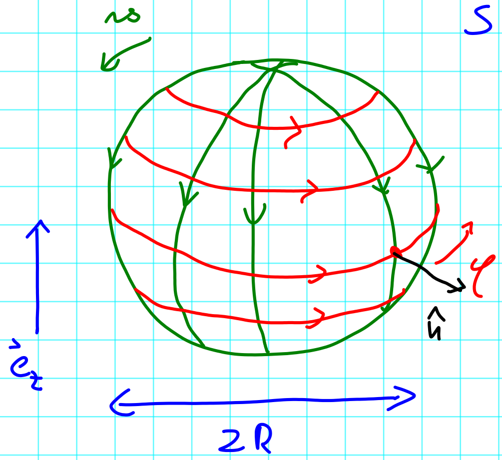
$$\varphi, z \longmapsto \vec S(\varphi, z) = \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix}$$

$$\frac{\partial \vec S}{\partial \varphi} = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix} = R\,\vec e_\varphi$$

$$\frac{\partial \vec S}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \vec e_z$$

$$\hat n = \frac{1}{|\cancel{R}\,\vec e_\varrho|} \cancel{R}\, \underbrace{\vec e_\varphi \times \vec e_z}_{\vec e_\varrho} = \vec e_\varrho$$

Sphäre (Kugeloberfläche)



$$S : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$$

$$\vartheta, \varphi \mapsto \vec{S}(\vartheta, \varphi) = R \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix}$$

$$\underline{\underline{\frac{\partial \vec{S}}{\partial \vartheta}}} = R \begin{pmatrix} \cos\varphi \cos\vartheta \\ \sin\varphi \cos\vartheta \\ -\sin\vartheta \end{pmatrix} = \underline{\underline{R}}\, \vec{e}_\vartheta$$

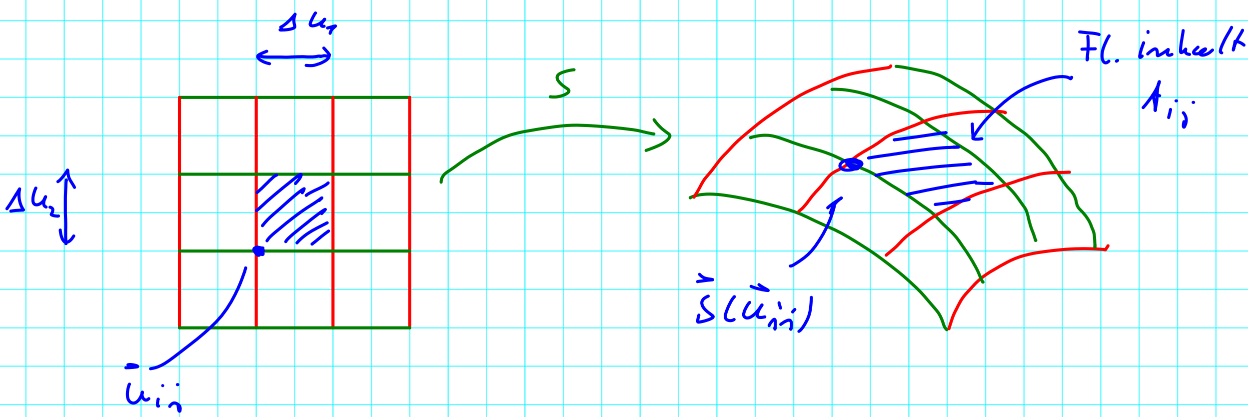
$$\underline{\underline{\frac{\partial \vec{S}}{\partial \varphi}}} = R \begin{pmatrix} -\sin\varphi \sin\vartheta \\ \cos\varphi \sin\vartheta \\ 0 \end{pmatrix}$$

$$= \underline{\underline{R \sin\vartheta}}\, \vec{e}_\varphi$$

$$\to \quad \hat{n} = \vec{e}_\vartheta \times \vec{e}_\varphi = \vec{e}_r$$

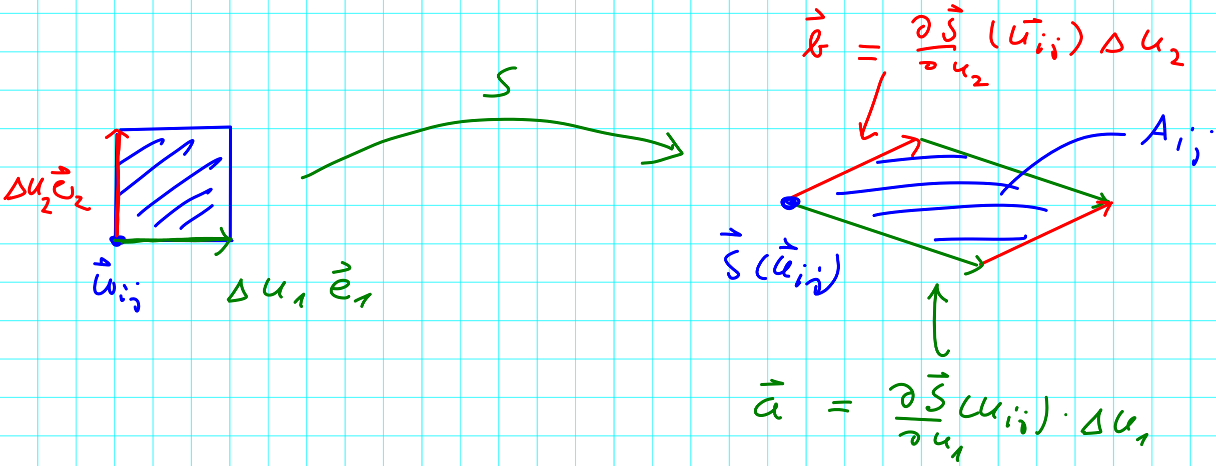
Flächenintegralen :

1. Flächeninhalt $A(S)$ eines Flächenstücks :



$$A(S) = \sum_{i,j} A_{ij}$$

Bestimmung von A_{ij}



$$A_{ij} = |\vec a \times \vec b| = \left| \frac{\partial \vec s}{\partial u_1} \times \frac{\partial \vec s}{\partial u_2} \right| \Delta u_1 \Delta u_2$$

$$\to \quad A(S) = \sum_{i,j} A_{ij} = \sum_{i,j} \left| \frac{\partial \vec s}{\partial u_1} \times \frac{\partial \vec s}{\partial u_2} \right| \Delta u_1 \Delta u_2$$

$$\to \quad \boxed{A(S) = \int_S |d\vec f| := \int_{a_1}^{b_1} \int_{a_2}^{b_2} \left| \frac{\partial \vec s}{\partial u_1}(\vec u) \times \frac{\partial \vec s}{\partial u_2}(\vec u) \right| du_1\, du_2}$$

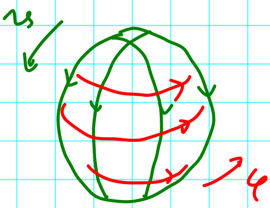
Notation

$$\oint_C \vec A\, d\vec \ell = \int_C \vec A\, d\vec \ell$$

$$\oiint_S \vec A\, d\vec \ell = \int_S \vec A\, d\vec f$$

$$\boxed{A(S) = \int_S |d\vec f| := \int_{a_1}^{b_1}\int_{a_2}^{b_2} \left| \frac{\partial \vec s}{\partial u_1}(\vec u) \times \frac{\partial \vec s}{\partial u_2}(\vec u)\right| du_1\, du_2}$$

Bsp: Flächeninhalt der Sphäre vom Radius R:



wie oben: $S : [0,\pi] \times [0, 2\pi] \longrightarrow \mathbb{R}$

$$\vartheta \times \varphi \longmapsto \vec s(\vartheta, \varphi)$$

$$\frac{\partial \vec s}{\partial \vartheta} = \underline{R}\, \vec e_\vartheta \quad , \quad \frac{\partial \vec s}{\partial \varphi} = \underline{R \sin\vartheta}\, \vec e_\varphi$$

$$\to \left| \frac{\partial \vec s}{\partial \vartheta} \times \frac{\partial \vec s}{\partial \varphi} \right| = R^2 \sin\vartheta \underbrace{|\vec e_\vartheta \times \vec e_\varphi|}_{1}$$

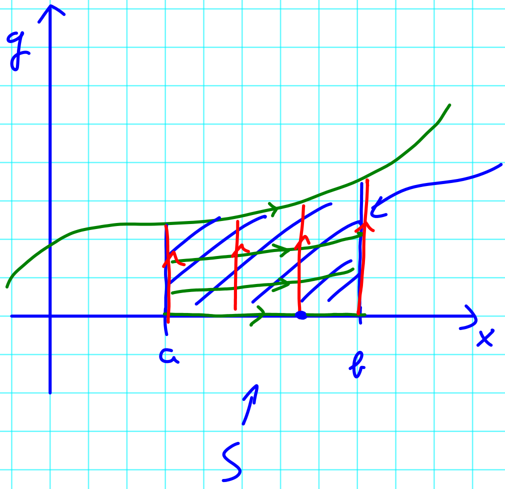
$$\to A(S) = \int_S |d\vec f| = \int_0^\pi \int_0^{2\pi} \underbrace{R^2 \sin\vartheta \, d\vartheta \, d\varphi}_{|d\vec f|:\ \text{Flächelement im sph.}}$$

$$= \int_0^\pi \Bigg(\underbrace{\int_0^{2\pi} \underline{R^2 \sin\vartheta}\, d\varphi}_{2\pi R^2 \sin\vartheta} \Bigg) d\vartheta$$

$$= 2\pi R^2 \int_0^\pi \sin\vartheta \, d\vartheta = 2\pi R^2 \Big(\underbrace{-\cos\vartheta \Big|_0^\pi}_{2} \Big) = 4\pi R^2$$

$$A(S) = 4\pi R^2$$

2)



$$A(S) = \int_a^b g(x)\,dx \quad !$$

Parametrisierung: $S : [a,b] \times [0,1] \longrightarrow \mathbb{R}^3$

$$(x,u) \longmapsto \vec{S}(x,u) = \begin{pmatrix} x \\ u\,g(x) \\ 0 \end{pmatrix}$$

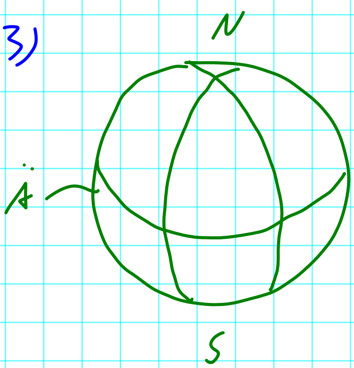
$$\frac{\partial \vec{S}}{\partial x} = \begin{pmatrix} 1 \\ u\,g'(x) \\ 0 \end{pmatrix}, \quad \frac{\partial \vec{S}}{\partial u} = \begin{pmatrix} 0 \\ g(x) \\ 0 \end{pmatrix}$$

$$\rightarrow \left| \frac{\partial \vec{S}}{\partial x} \times \frac{\partial \vec{S}}{\partial u} \right| = \left| \begin{pmatrix} 0 \\ 0 \\ g(x) \end{pmatrix} \right| = |g(x)|$$

$$\rightarrow A(S) = \int_a^b \underbrace{\int_0^1 |g(x)|\,du}_{g(x)}\,dx = \int_a^b g(x)\,dx \quad \checkmark$$

$g(x) > 0$

3)

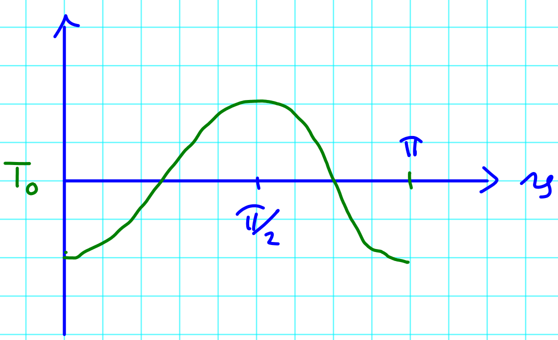


$S : [0,\pi] \times [0,2\pi] \longrightarrow \mathbb{R}^3$

$(\vartheta, \varphi) \longmapsto \vec{S}(\vartheta, \varphi)$

Temperatur $T(\vartheta, \cancel{\varphi}) = T(\vartheta)$

Annahme: $T(\vartheta) = T_0 - \frac{\Delta T}{2} \cos(2\vartheta)$



mittlere Temperatur:

$\bar{T} = ?$

$$\overline{T} = \frac{1}{4\pi R^2} \int_S T \, |d\vec{f}|$$

$$= \frac{1}{4\pi R^2} \int_0^{\pi} \underbrace{\int_0^{2\pi} T(\vartheta) \cdot R^2 \sin\vartheta \, d\varphi}_{2\pi \, T(\vartheta) \, R^2 \sin\vartheta} \, d\vartheta$$

$$= \frac{1}{2} \int_0^{\pi} T(\vartheta) \sin\vartheta \, d\vartheta$$

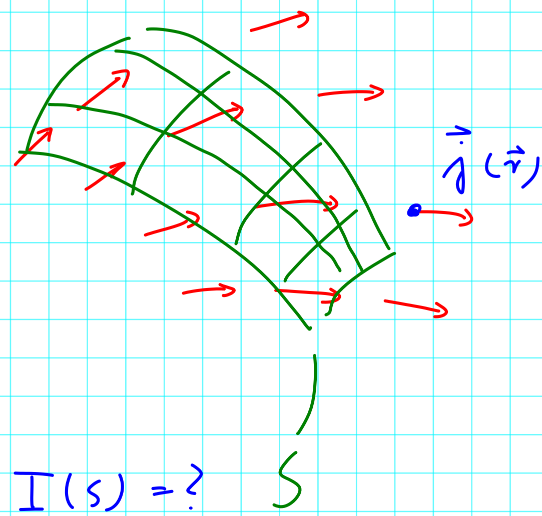
$$= \frac{1}{2} \int_0^{\pi} \left(T_0 - \frac{\Delta T}{2} \cos(2\vartheta) \right) \sin\vartheta \, d\vartheta$$

$$= \underbrace{\frac{1}{2} \int_0^{\pi} T_0 \sin\vartheta \, d\vartheta}_{T_0} - \frac{\Delta T}{4} \underbrace{\int_0^{\pi} \cos(2\vartheta) \sin\vartheta \, d\vartheta}_{-2/3}$$

$$\rightarrow \quad \overline{T} = T_0 + \frac{\Delta T}{6}$$

3) Flächenintegral eines Vektorfeldes über ein Flächenstück S:

Strömung



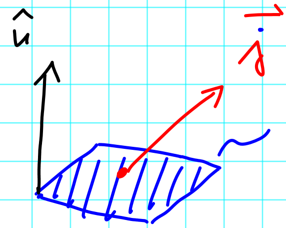
→ Stromdichte

$\vec{j} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$\vec{r} \mapsto \vec{j}(\vec{r})$

↳ Strom durch Fl. S: $I(S) = ?$

Stromdichte:



$\Delta A \quad (\to 0)$

$$\boxed{\Delta I = I(\Delta A) := \langle \vec{j}, \hat{n} \rangle \, \Delta A}$$

mit <u>Flächenelementvektor</u> $\quad \Delta \vec{f} := \Delta A \cdot \hat{n}$

$$\rightsquigarrow \quad \Delta I = \langle \vec{j}, \Delta \vec{f} \rangle$$

$$\rightsquigarrow \quad I(S) = \sum_{ij} \Delta I_{ij}$$

$$\Delta I_{ij} = \Big\langle \vec{j}(\vec{s}(\vec{u}_{ij})), \underbrace{\frac{\frac{\partial \vec{s}}{\partial u_1} \times \frac{\partial \vec{s}}{\partial u_2}}{\cancel{\left| \frac{\partial \vec{s}}{\partial u_1} \times \frac{\partial \vec{s}}{\partial u_2} \right|}}}_{\hat{n}_{ij}} \Big\rangle \cdot \underbrace{\cancel{\left| \frac{\partial \vec{s}}{\partial u_1} \times \frac{\partial \vec{s}}{\partial u_2} \right|}}_{\Delta A_{ij}} \Delta u_1 \Delta u_2$$

$$\hookrightarrow \quad I(S) = \sum_{ij} \Big\langle \underbrace{\vec{j}(\vec{s}(\vec{u}_{ij}))}_{\vec{j}}, \underbrace{\frac{\partial \vec{s}}{\partial u_1} \times \frac{\partial \vec{s}}{\partial u_2} \Big\rangle \Delta u_1 \Delta u_2}_{d\vec{f}}$$

$$\rightsquigarrow \quad \boxed{I(S) = \int_S \vec{j} \, d\vec{f} := \int_{a_1}^{b_1} \int_{a_2}^{b_2} \Big\langle \vec{j}(\vec{s}(\vec{u})), \frac{\partial \vec{s}}{\partial u_1} \times \frac{\partial \vec{s}}{\partial u_2} \Big\rangle \, du_2 \, du_1}$$