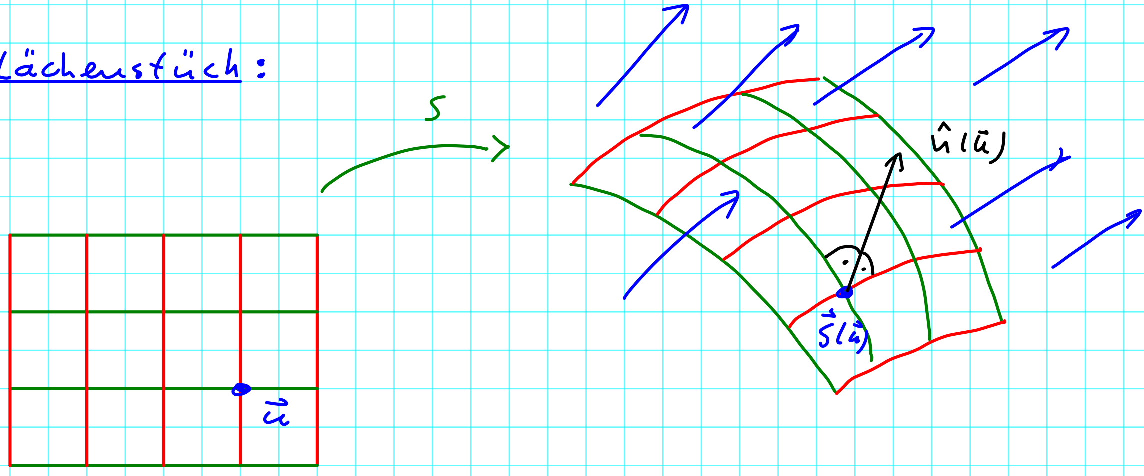
Flächenstück:




Parametrisierung: $S : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}^3$

$$\vec{u} = (u_1, u_2) \mapsto \vec{S}(\vec{u})$$

$\to$ Flächennormale: $$\hat{n} = \frac{1}{\left| \frac{\partial \vec{S}}{\partial u_1} \times \frac{\partial \vec{S}}{\partial u_2} \right|} \frac{\partial \vec{S}}{\partial u_1} \times \frac{\partial \vec{S}}{\partial u_2}$$

inf. Flächenelement:

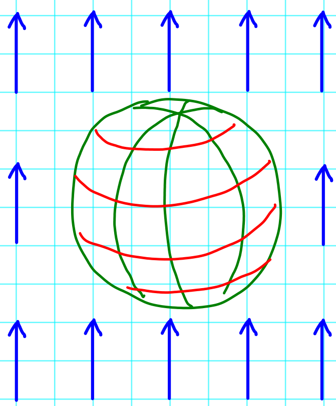
$$d\vec{f} = \frac{\partial \vec{S}}{\partial u_1} \times \frac{\partial \vec{S}}{\partial u_2} \, du_1 du_2$$

$\to$ • $$\boxed{A(S) = \int_S |d\vec{f}| = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \left| \frac{\partial \vec{S}}{\partial u_1} \times \frac{\partial \vec{S}}{\partial u_2} \right| du_1 du_2}$$

• $$\int_S T \, |d\vec{f}| = \int_{a_1}^{b_1} \int_{a_2}^{b_2} T(\vec{u}) \left| \frac{\partial \vec{S}}{\partial u_1} \times \frac{\partial \vec{S}}{\partial u_2} \right| du_1 du_2$$

• $$\int_S \vec{j} \, d\vec{f} = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \left\langle \vec{j}(\vec{S}(\vec{u})), \frac{\partial \vec{S}}{\partial u_1} \times \frac{\partial \vec{S}}{\partial u_2} \right\rangle du_1 du_2$$

Beispiel: Strom $I(S)$ durch Sphäre $S$



$$S : [0,\pi] \times [0, 2\pi] \to \mathbb{R}^3$$
$$(\vartheta, \varphi) \mapsto \vec{s}(\vartheta,\varphi) = \cdots$$

$$\boxed{\vec{j} = j_0 \, \vec{e}_3}$$

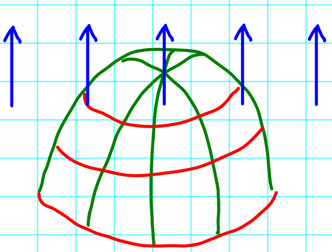
$$\boxed{d\vec{f} = R^2 \sin\vartheta \; \vec{e}_r \; d\varphi \, d\vartheta}$$

$$\to \quad I(S) = \int_S \vec{j} \, d\vec{f} = \int_0^{\pi} \int_0^{2\pi} j_0 \underbrace{\langle \vec{e}_3, \vec{e}_r \rangle}_{= \cos\vartheta} R^2 \sin\vartheta \, d\varphi \, d\vartheta$$

$$= 2\pi R^2 j_0 \int_0^{\pi} \cos\vartheta \sin\vartheta \, d\vartheta$$

$$= 2\pi R^2 j_0 \left( \tfrac{1}{2} \sin^2\vartheta \Big|_0^{\pi} \right) = 0$$

Strom $I(H)$ durch Halbsphäre:



$$H : [0, \tfrac{\pi}{2}] \times [0, 2\pi] \to \mathbb{R}^3$$
$$(\vartheta, \varphi) \mapsto \vec{s}(\vartheta,\varphi) = \cdots$$

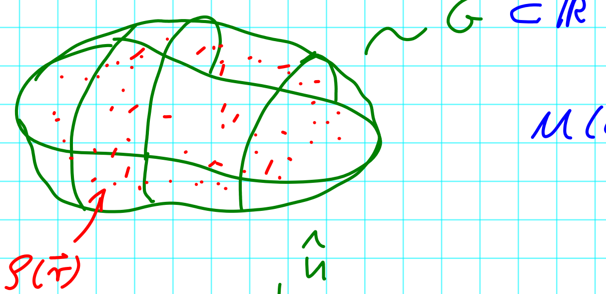
$$I(H) = 2\pi R^2 j_0 \int_0^{\pi/2} \cos\vartheta \sin\vartheta \, d\vartheta$$

$$= 2\pi R^2 j_0 \underbrace{\left( \tfrac{1}{2} \sin^2\vartheta \Big|_0^{\pi/2} \right)}_{= 1/2} = \pi R^2 \cdot j_0 \quad \checkmark$$

# Volumengebiete und Volumenintegrale

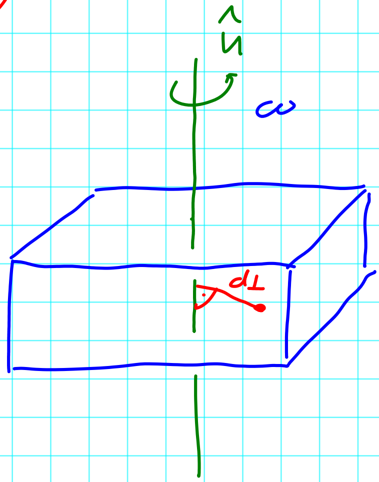
Motivation:

a)




$$M(G) \stackrel{!}{=} \int_G \rho \, dV$$

b)




Rotationsenergie

$$E_{Rot} = \frac{1}{2} I_{\hat{n}} \omega^2$$

$$L_{\hat{n}} = I_{\hat{n}} \cdot \omega$$

Trägheitsmoment: $\quad I_{\hat{n}} \stackrel{!}{=} \int_G \rho \, d_\perp^2 \, dV$

## Parametrisierung eines Volumengebiets $G$:

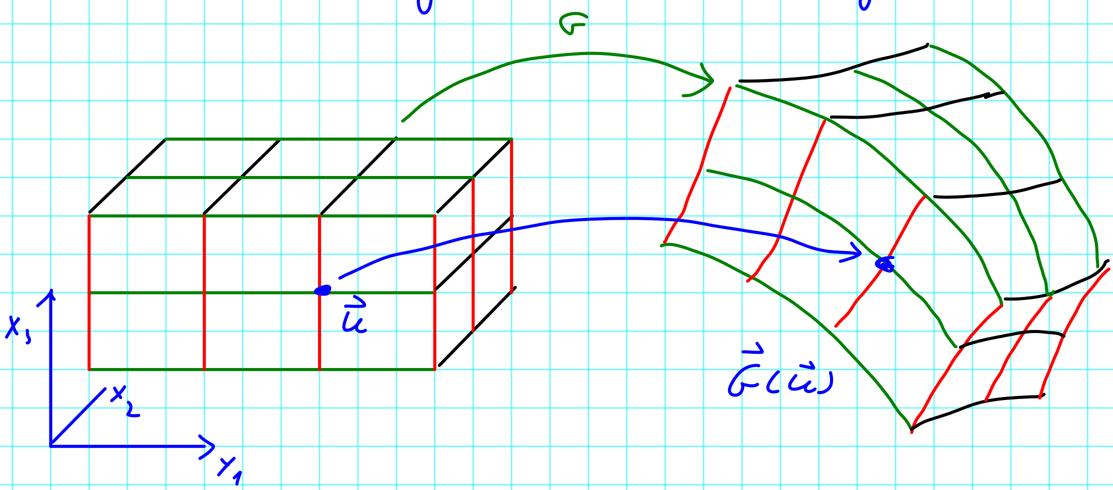
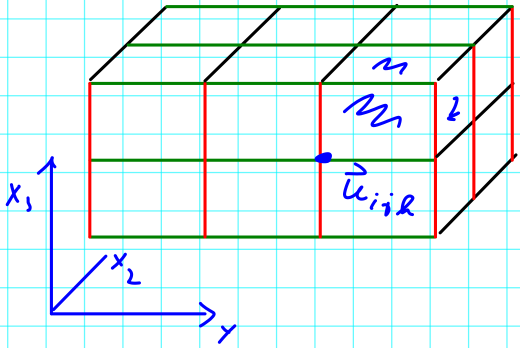
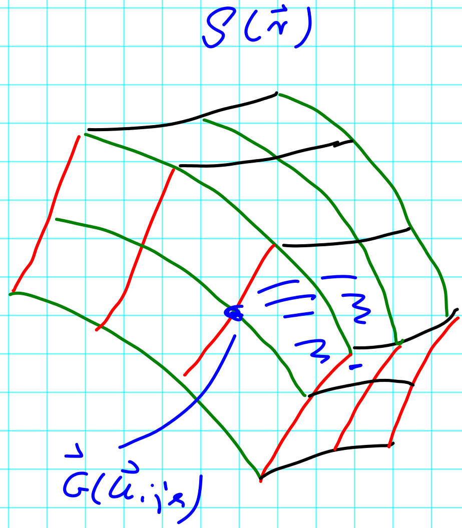



Abb: $\quad G : [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \to \mathbb{R}^3$

$$\vec{u} = (u_1, u_2, u_3) \longmapsto \vec{G}(\vec{u})$$

Volumenintegral




$\Delta V_{ijk}$

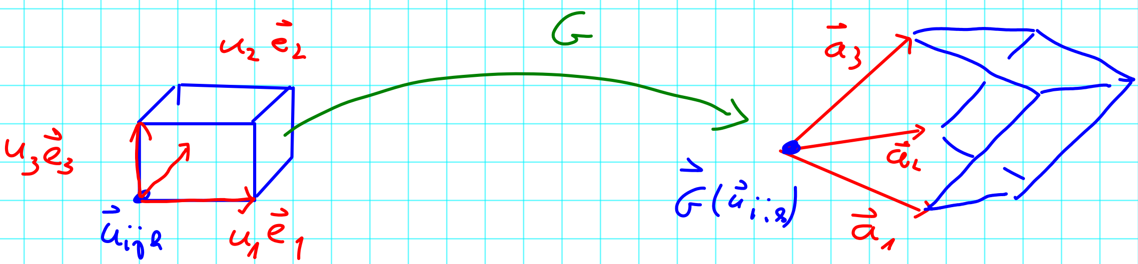
$$\Delta M_{ijk} = \varrho_{ijk} \, \Delta V_{ijk}$$

$$M(G) = \sum_{ijk} \Delta M_{ijk}$$

$$= \sum_{ijk} \varrho_{ijk} \, \Delta V_{ijk}$$

$$= \varrho(\vec{G}(\vec{u}_{ijk}))$$

Bestimmung von $V_{ijk}$




$$\Delta V_{ijk} = \left| \langle \vec{a}_1 \times \vec{a}_2, \vec{a}_3 \rangle \right|$$

mit $\vec{a}_i = \dfrac{\partial \vec{G}}{\partial u_i} u_i$ erhalten wir

$$\Delta V_{ijk} = \left| \left\langle \frac{\partial \vec{G}}{\partial u_1} \times \frac{\partial \vec{G}}{\partial u_2}, \frac{\partial \vec{G}}{\partial u_3} \right\rangle \right| u_1 u_2 u_3$$

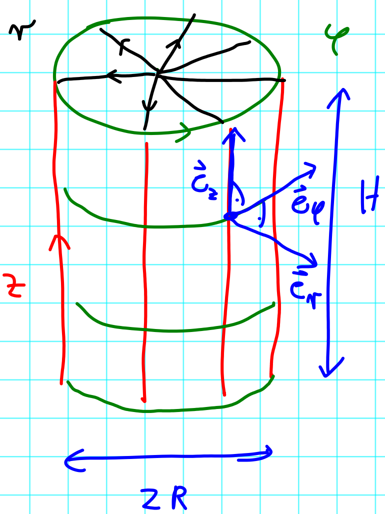
$$\rightarrow \boxed{\begin{array}{l} M(G) = \displaystyle\sum_{ijk} \varrho_{ijk} \cdot \left| \left\langle \frac{\partial \vec{G}}{\partial u_1} \times \frac{\partial \vec{G}}{\partial u_2}, \frac{\partial \vec{G}}{\partial u_3} \right\rangle \right| \Delta u_1 \Delta u_2 \Delta u_3 \\ \Delta u_1, \Delta u_2, \Delta u_3 \rightarrow 0 \end{array}}$$

$$\boxed{M(G) = \int_G \rho \, dV := \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \rho(\vec{G}(\vec{u})) \left| \left\langle \frac{\partial \vec{G}}{\partial u_1} \times \frac{\partial \vec{G}}{\partial u_2}, \frac{\partial \vec{G}}{\partial u_3} \right\rangle \right| du_1 \, du_2 \, du_3}$$

$$\int_G \rho \, dV = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \rho(\vec{G}(\vec{u})) \left| \det\left( \frac{\partial \vec{G}}{\partial u_1}, \frac{\partial \vec{G}}{\partial u_2}, \frac{\partial \vec{G}}{\partial u_3} \right) \right| du_1 \, du_2 \, du_3$$

$d=3$ $\quad \langle \vec{a} \times \vec{b}, \vec{c} \rangle = \det(\vec{a}, \vec{b}, \vec{c})$

<u>Beispiel</u>: Zylinder




$$G: [0, 2\pi] \times [0, H] \times [0, R] \longrightarrow \mathbb{R}^3$$

$$(\varphi, z, r) \longmapsto \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$$

$$\frac{\partial \vec{G}}{\partial \varphi} = r \underbrace{\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}}_{\vec{e}_\varphi} = r \, \vec{e}_\varphi$$

$$\frac{\partial \vec{G}}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \vec{e}_z$$

$$\frac{\partial \vec{G}}{\partial r} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = \vec{e}_r$$

$$dV = \overbrace{\left| \left\langle \frac{\partial \vec{G}}{\partial \varphi} \times \frac{\partial \vec{G}}{\partial z}, \frac{\partial \vec{G}}{\partial r} \right\rangle \right|}^{r \,=} d\varphi \, dz \, dr$$

$$dV = r \, d\varphi \, dz \, dr$$

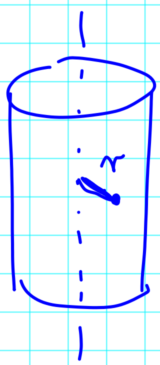
Volumen: $V = \int_G 1\,dV =$

$$= \int_0^{2\pi}\int_0^{H}\int_0^{R} r\,d\varphi\,dz\,dr$$

$$= \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\;\underbrace{\int_0^{H} dz}_{H}\;\underbrace{\int_0^{R} r\,dr}_{R^2/2} = \pi R^2 H$$

Trägheitsmoment $\qquad \varrho_0 = \text{konst}$

$$I_3 = \int_G \varrho\, r^2\,dV = \int_0^{2\pi}\int_0^{H}\int_0^{R} \varrho_0\, r^2 \cdot r\,d\varphi\,dz\,dr$$




$r = d_\perp$

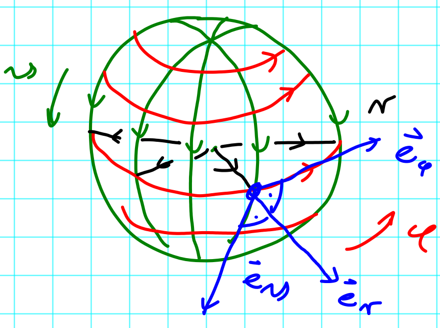
$$= 2\pi H \varrho_0 \underbrace{\int_0^{R} r^3\,dr}_{R^4/4}$$

$$= \frac{1}{2}\pi \underbrace{R^4 H \varrho_0}_{R^2 \underbrace{\mathrm{Vol}\ \varrho_0}_{M}} = \frac{1}{2} M R^2$$

2) Kugel

$$S : [0, \pi] \times [0, 2\pi] \times [0, R] \to \mathbb{R}^3$$

$$(\vartheta, \varphi, r) \mapsto r \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix}$$




$$\frac{\partial \vec{S}}{\partial \vartheta} = r\,\vec{e}_\vartheta, \quad \frac{\partial \vec{S}}{\partial \varphi} = r \sin\vartheta\,\vec{e}_\varphi, \quad \frac{\partial \vec{S}}{\partial r} = \vec{e}_r$$

$$\rightsquigarrow dV = \left| \left\langle \frac{\partial \vec{s}}{\partial \vartheta} \times \frac{\partial \vec{s}}{\partial \varphi}, \frac{\partial \vec{s}}{\partial r} \right\rangle \right| d\vartheta\, d\varphi\, dr$$

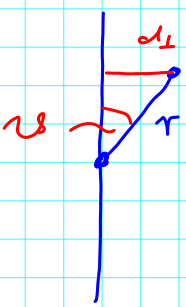
$$\boxed{dV = r^2 \sin\vartheta \, d\vartheta\, d\varphi\, dr}$$

Volumen der Kugel

$$V = \int_G 1\, dV = \int_0^\pi \underbrace{\int_0^{2\pi} \int_0^R r^2 \sin\vartheta\, d\vartheta\, d\varphi\, dr}_{= 4\pi r^2}$$

$$= 4\pi \int_0^R r^2\, dr = \frac{4}{3}\pi R^3$$

Trägheitsmoment: $\varrho_0 = \text{konst.}$




$$\rightarrow \boxed{d_\perp = r \sin\vartheta}$$

$$I = \int_G \varrho_0\, d_\perp^2\, dV$$

$$= \varrho_0 \int_0^\pi \int_0^{2\pi} \int_0^R \underbrace{r^2 \sin^2\vartheta}_{d_\perp^2}\ \underbrace{r^2 \sin\vartheta\, d\vartheta\, d\varphi\, dr}_{= dV}$$

$$= 2\pi \varrho_0 \underbrace{\int_0^R r^4\, dr}_{R^5/5} \underbrace{\int_0^\pi \sin^3\vartheta\, d\vartheta}_{= 4/3} = \frac{8\pi R^5}{3\cdot 5}\varrho_0$$

$$= \frac{2}{5} M R^2$$

$$M = \frac{4}{3}\pi R^3 \cdot \varrho_0$$