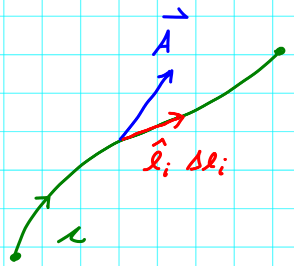
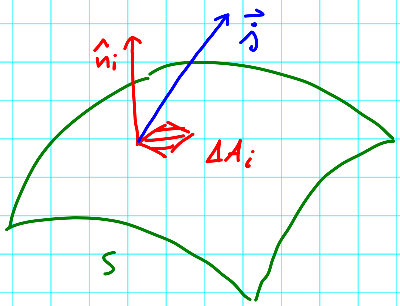
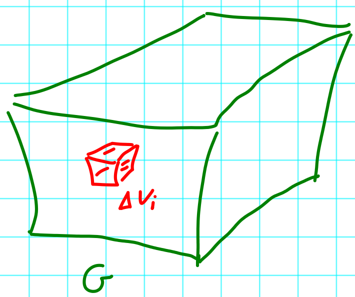
## Linien / Flächen / Volumen - Integrale:




$$\int_L \vec A\, d\vec\ell := \sum_i \langle \vec A_i, \hat\ell_i \rangle\, \Delta\ell_i = \dots$$




$$\int_S \vec j\, d\vec f := \sum_i \langle \vec j_i, \hat n_i \rangle\, \Delta A_i = \dots$$

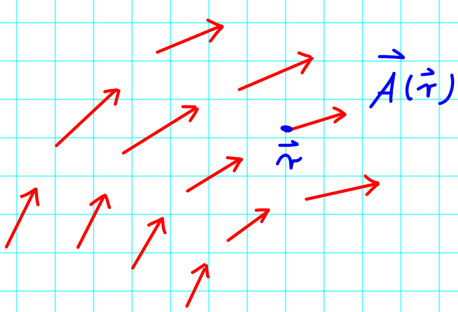



$$\int_G \rho\, dV := \sum_i \rho_i\, \Delta V_i = \dots$$

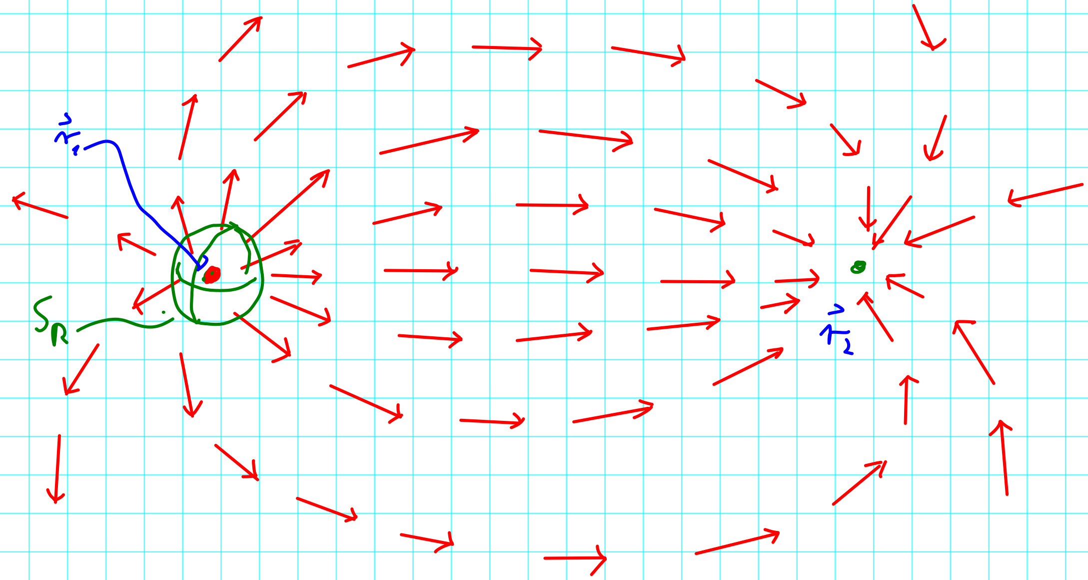
heute: <u>Divergenz (= Quellstärke) eines Vektorfelds,</u>
<u>Satz von Gauß</u>

Vektorfeld $\vec A$ $\rightsquigarrow$ Divergenz/Quellstärke von $\vec A$ am Ort $\vec r$:

$$\operatorname{div} \vec A(\vec r) = ?$$



Beispiel: Stromdichte $\vec{j}$ einer stat. Strömung




Wo befinden sich Quellen / Senken der Strömung?

qualitativ: $\quad \operatorname{div} \vec{j}(\vec{r}_1) \overset{!}{>} 0 \quad ; \quad \operatorname{div} \vec{j}(\vec{r}_2) \overset{!}{<} 0$

quantitativ:

$S_R$: Sphäre vom Radius $R$ um $\vec{r}_1$:

Idee:

$$\operatorname{div} \vec{j}(\vec{r}_1) \overset{!}{=} \lim_{R \to 0} \frac{1}{\mathrm{Vol}(K_R)} \underbrace{\int_{S_R} \vec{j}\, d\vec{f}}_{> 0 \ \checkmark}$$

allgemeiner:

**Def.**: Divergenz des Vf. $\vec{A}$ im Pkt. $\vec{r}$:

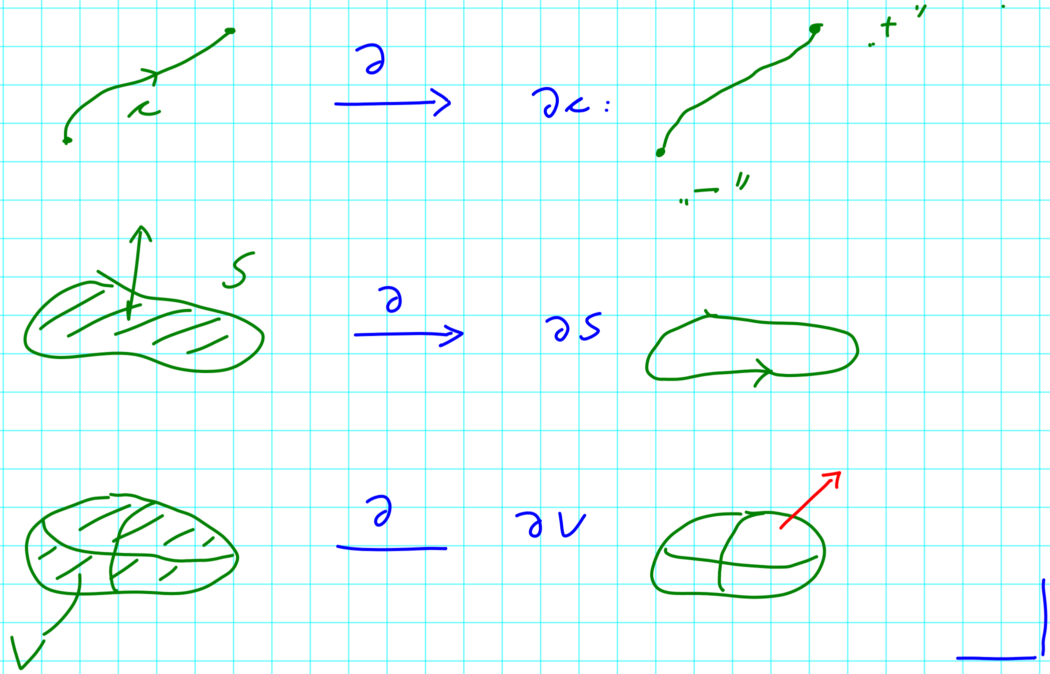
$$\operatorname{div} \vec{A}(\vec{r}) := \lim_{|V| \to 0} \frac{1}{|V|} \int_{\partial V} \vec{j}\, d\vec{f}$$

wobei: $V$: Volumengebiet um $\vec{r}$

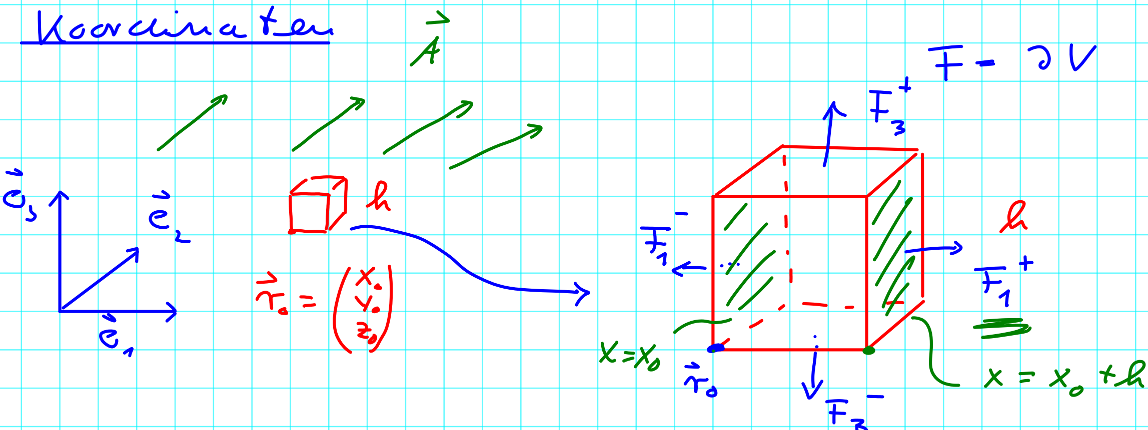
$|V|$: Volumeninhalt von $V$

$\partial V$: Oberfläche von $V$

$\partial$: "Randoperator":




## Berechnung der Divergenz in kartesischen Koordinaten




d.h. $F = F_1^+ \cup F_1^- \cup F_2^+ \cup F_2^- \cup F_3^+ \cup F_3^-$

$$\to \int_F \vec A \, d\vec f = \underbrace{\int_{F_1^+ \cup F_1^-} \vec A \, d\vec f}_{I_1} + \underbrace{\int_{F_2^+ \cup F_2^-} \vec A \, d\vec f}_{I_2} + \underbrace{\int_{F_3^+ \cup F_3^-} \vec A \, d\vec f}_{I_3}$$

$$\underline{\underline{I_1}} = \int_{y_0}^{y_0+h} dy \int_{z_0}^{z_0+h} dz \left\{ \underbrace{A_1(x_0+h, y, z) - A_1(x_0, y, z)}_{\underset{h \to 0}{=} h \frac{\partial A_1}{\partial x_1}(\vec r_0)} \right\} = \underline{\underline{h^3 \frac{\partial A_1}{\partial x_1}(\vec r_0)}}$$

analog: $$I_2 = h^3 \frac{\partial A_2}{\partial x_2}(\vec{r}_0)$$

$$I_3 = h^3 \frac{\partial A_3}{\partial x_3}(\vec{r}_0)$$

$$\to \operatorname{div} \vec{A}(\vec{r}_0) = \lim_{h\to 0} \frac{1}{\cancel{h^3}}\left(\cancel{h^3}\frac{\partial A_1}{\partial x_1} + \cancel{h^3}\frac{\partial A_2}{\partial x_2} + \cancel{h^3}\frac{\partial A_3}{\partial x_3}\right)_{\vec{r}_0}$$

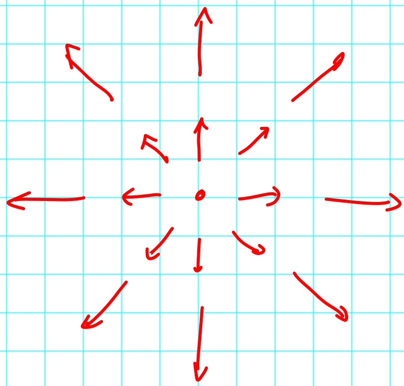
d.h. $$\boxed{\operatorname{div} \vec{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}}$$

$$= \sum_{i=1}^{3} \frac{\partial A_i}{\partial x_i}$$

$$= \vec{\nabla} \cdot \vec{A}, \quad \vec{\nabla} = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix}$$

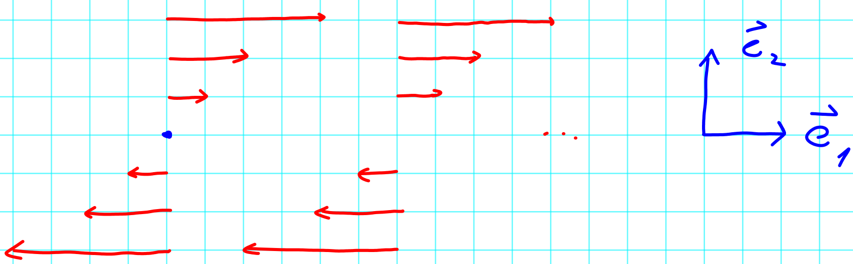
<u>Beispiele</u>:

1) $\vec{A}(\vec{r}) = \vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$



$$\operatorname{div} \vec{A} = \operatorname{div} \vec{r} = \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} + \frac{\partial x_3}{\partial x_3} = 3$$

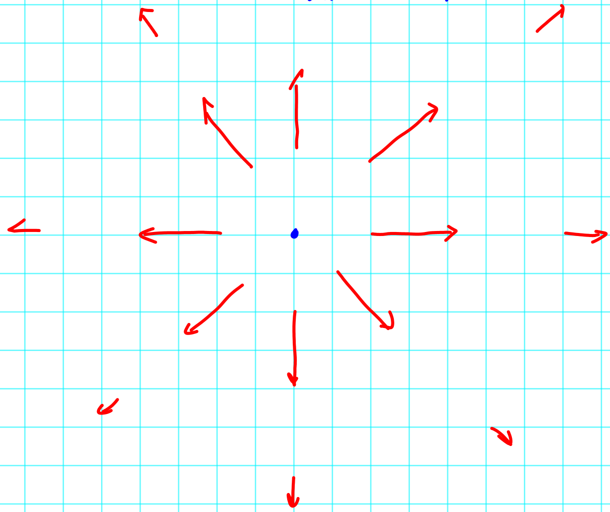
2) $\vec{A}(\vec{r})$ :




$$\vec{A}(\vec{r}) = x_2 \vec{e}_1 = \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix}$$

$$\operatorname{div} \vec{A}(\vec{r}) = \frac{\partial x_2}{\partial x_1} + \frac{\partial 0}{\partial x_2} + \frac{\partial 0}{\partial x_3} = 0$$

3) Stromdichte: $\vec{j}(\vec{r}) = \frac{I_0}{4\pi} \frac{\hat{r}}{r^2}$



Divergenz für $\underline{\vec{r} \neq 0}$

$$\mathrm{div} \frac{\hat{r}}{r^2} = \mathrm{div} \frac{\vec{r}}{r^3} = \mathrm{div} \frac{1}{r^3} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

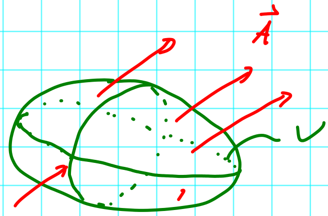
$$= \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( \frac{1}{r^3} x_i \right) = \sum_{i=1}^{3} \left( \frac{1}{r^3} + x_i \frac{\partial}{\partial x_i} \frac{1}{r^3} \right)$$

N.R.: $\frac{\partial}{\partial x_1} (x_1^2 + x_2^2 + x_3)^{-3/2} = -\frac{3}{\cancel{2}} \frac{1}{r^5} \cancel{2} x_1 = -\frac{3}{r^5} x_1$

analog: $\frac{\partial}{\partial x_i} \frac{1}{r^3} = -3 \frac{x_i}{r^5}$ (*)

$$\rightarrow \mathrm{div} \frac{\hat{r}}{r^2} = \sum_{i=1}^{3} \left( \frac{1}{r^3} - \frac{3 x_i^2}{r^5} \right) = \frac{3}{r^3} - 3 \underbrace{\frac{\sum_i x_i^2}{r^5}}_{= 1/r^3} = 0 \,!$$

# Satz von Gauß




$$\boxed{\int_{\partial V} \vec{A}\, d\vec{f} \overset{!}{=} \int_{V} \operatorname{div} \vec{A}\; dV}$$