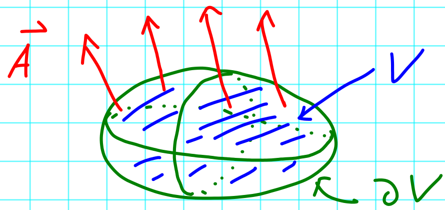
Wiederholung: Divergenz (Quellstärke) eines Vf.s:

$$\boxed{\operatorname{div} \vec{A} := \lim_{|V| \to 0} \frac{1}{|V|} \int_{\partial V} \vec{A}\, d\vec{f}}$$

$$= \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} \qquad (= \vec{\nabla} \cdot \vec{A})$$

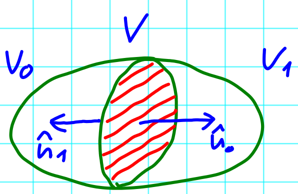
Satz von Gauß



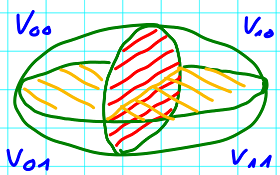
$$\boxed{\int_{\partial V} \vec{A}\, d\vec{f} = \int_V \operatorname{div} \vec{A}\, dV}$$

Fluss von $\vec{A}$ durch ∂V — Gesamtquellstärke von $\vec{A}$ in V

"Physikerbeweis":

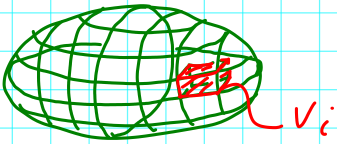


$$\int_{\partial V} \vec{A}\, d\vec{f} \overset{!}{=} \int_{\partial V_0} \vec{A}\, d\vec{f} + \int_{\partial V_1} \vec{A}\, d\vec{f}$$



$$\overset{!}{=} \int_{\partial V_{00}} \vec{A}\, d\vec{f} + \int_{\partial V_{01}} \vec{A}\, d\vec{f} + \int_{\partial V_{10}} \vec{A}\, d\vec{f} + \int_{\partial V_{11}} \vec{A}\, d\vec{f}$$

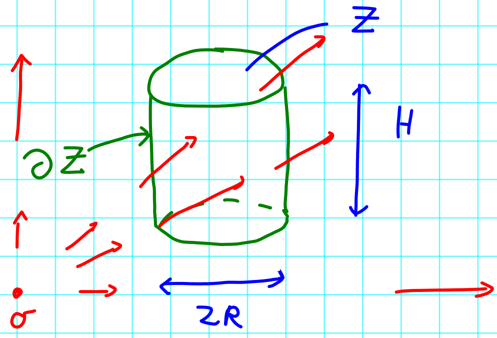
⋮



$$= \underbrace{\sum_i}_{} \underbrace{\frac{1}{|V_i|} \int_{\partial V_i} \vec{A}\, d\vec{f}}_{} \underbrace{|V_i|}_{} \qquad |V_i| \to 0$$

$$= \int_V \operatorname{div} \vec{A} \qquad dV$$

Anwendungsbeispiel

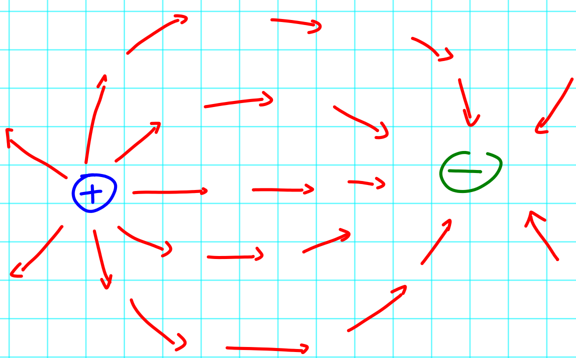


$\vec A(\vec r) = \vec r$

$$\int_{\partial Z} \vec A \, d\vec f \overset{\text{Gauß}}{=} \int_{Z} \operatorname{div} \vec A \, dV \underset{\operatorname{div}\vec r = 3}{=} 3 \int_{Z} dV = 3 \pi R^2 H$$

Elektrostatik:

„elektrische Ladungen sind Quellen des elektrischen Felds"

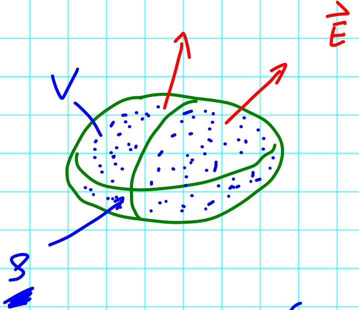


$$\boxed{\operatorname{div} \vec E = \rho / \varepsilon_0}$$

Gaußsches Gesetz

q

$\rightsquigarrow \vec E(\vec r)$?



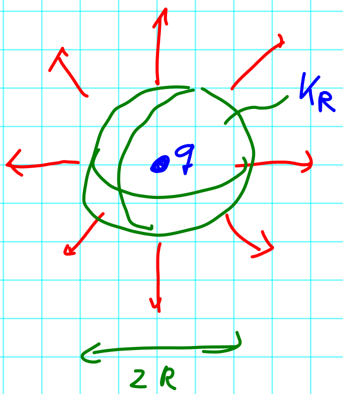
$$\rho/\varepsilon_0 = \operatorname{div}\vec E$$

$$Q_V = \int_V \rho \, dV = \varepsilon_0 \int_V \operatorname{div}\vec E \, dV \overset{\text{S.v.G.}}{=} \varepsilon_0 \int_{\partial V} \vec E \, d\vec f$$

d.h. $$\boxed{\int_{\partial V} \vec E \, d\vec f = \frac{1}{\varepsilon_0} Q_V} \quad ! \quad (*)$$

Gaußsches Gesetz in integraler Form

Bsp.: Elekt. Feld einer Pkt Ldg q im 0:



Ansatz: $\vec E(\vec r) \overset{?}{=} E(r)\,\hat r$

$$q \overset{!}{=} Q_R \overset{(*)}{=} \varepsilon_0 \int_{\partial K_R} \vec E \, d\vec f$$

$$= \varepsilon_0 E(R) \underbrace{\int_{S_R} |d\vec f|}_{= 4\pi R^2} = 4\pi R^2 \varepsilon_0 E(R)$$

$$\rightarrow E(R) = \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{R^2} \quad \leadsto \quad \vec E(\vec r) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \cdot \hat r$$

$$\vec r \neq 0: \quad \rho/\varepsilon_0 = \operatorname{div}\vec E = \frac{1}{4\pi\varepsilon_0} q \underbrace{\operatorname{div}\frac{\hat r}{r^2}}_{=0!} = 0 \; !$$

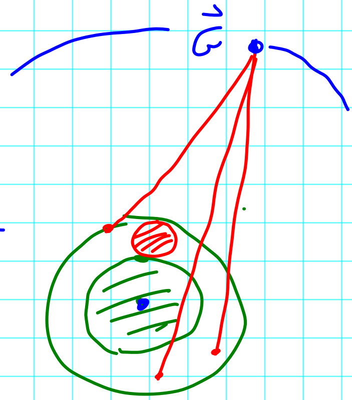
Newtonsche Gravitation:

$$\boxed{\operatorname{div}\vec G = -4\pi\gamma \rho_m}$$

Gravitations-feld

Massendichte

γ: Gravit. konstante

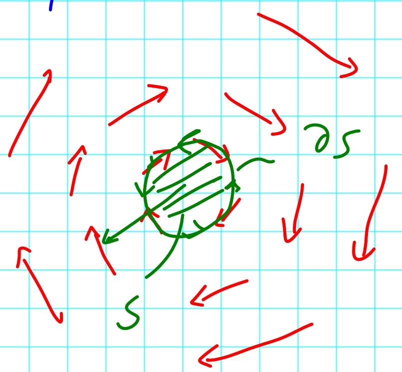


S.v. Gauß

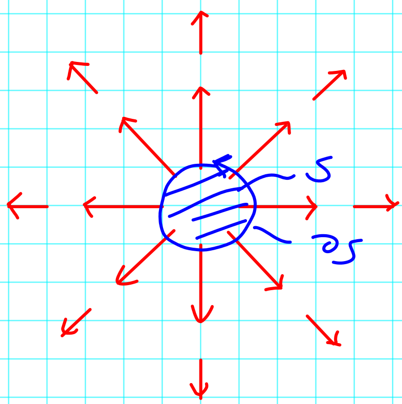
$$\boxed{\int_{\partial V} \vec G \, d\vec f = -4\pi\gamma M_V}$$

Rotation (Wirbelstärke) eines Vfes, Satz von Stokes

Vf. $\vec{A}$:



Vf. $\vec{B}$:



$$\oint_{\partial S} \vec{A}\, d\vec{\ell} \neq 0 \quad !$$

$$\oint_{\partial S} \vec{B}\, d\vec{\ell} = 0$$

Def.: Rotation (Wirbelstärke) des Vfs. $\vec{A}$ im Punkt $\vec{r}_0$

:= Vektor $\mathrm{rot}\,\vec{A}(\vec{r}_0)$ bestimmt durch:

$$\langle \hat{n}, \mathrm{rot}\,\vec{A}(\vec{r}_0) \rangle := \lim_{|S| \to 0} \frac{1}{|S|} \int_{\partial S} \vec{A}\, d\vec{\ell}$$

wobei:

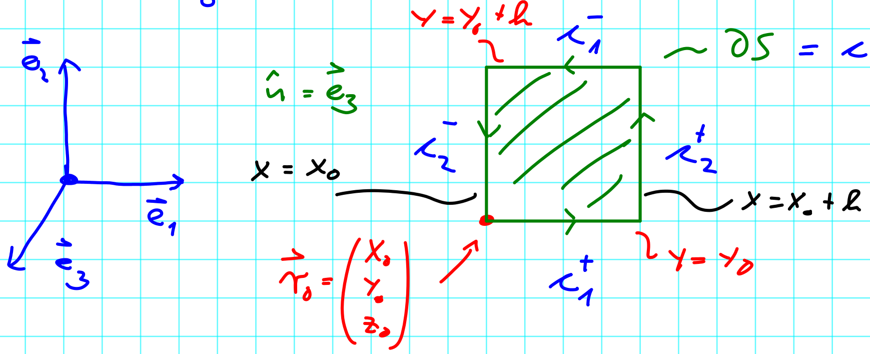
S : Flächenstück

$\hat{n}$: Flächennormale

$|S|$: Flächeninhalt

∂S : Rand der Fläche

Berechnung der Rotation in kart. Koordinaten:



$\sim \partial S = c = c_1^+ \cup c_1^- \cup c_2^+ \cup c_2^-$

l: Kantenlänge

$$\int_{c=\partial S} \vec A \, d\vec l = I_1 + I_2 ,$$

mit $$I_1 = \int_{c_1^+ \cup c_1^-} \vec A \, d\vec l = \int_{x_0}^{x_0+l} \underbrace{\left\{ A_1(x, y_0, z_0) - A_1(x, y_0 + l, z_0) \right\}}_{= -l \frac{\partial A_1}{\partial x_2}(\vec r_0)} dx$$

$\rightarrow \; I_1 = -l^2 \frac{\partial A_1}{\partial x_2}(\vec r_0)$

analog: $$I_2 = + \frac{\partial A_2}{\partial x_1}(\vec r_0) \, l^2$$

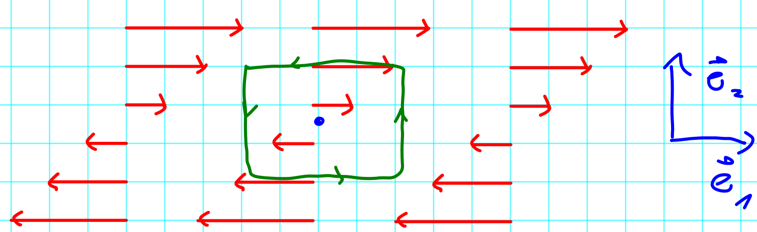
$$\left(\text{rot}\, \vec A(\vec r_0)\right)_3 = \langle \vec e_3, \text{rot}\, \vec A(\vec r_0) \rangle = \lim_{l \to 0} \frac{1}{\cancel{l^2}} \left(\cancel{l^2} \frac{\partial A_2}{\partial x_1}(\vec r_0) - \cancel{l^2} \frac{\partial A_1}{\partial x_2}(\vec r_0) \right)$$

$$= \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = \left(\vec\nabla \times \vec A \right)_3$$

$$\boxed{\text{rot}\, \vec A = \begin{pmatrix} \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \\ \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \\ \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \end{pmatrix} = \vec\nabla \times \vec A}$$

<u>Bsp.:</u>

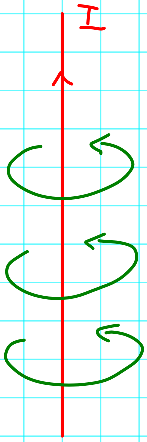
$$\vec j(\vec r) = x_2 \, \vec e_1 = \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix}$$



$$\text{rot}\, \vec j(\vec r) = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix} \times \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Anwendungen:

1) Magnetostatik: "elektrische Ströme erzeugen magnetische Wirbelfelder"



$$\boxed{\operatorname{rot} \vec{B} = \mu_0 \vec{j}}$$

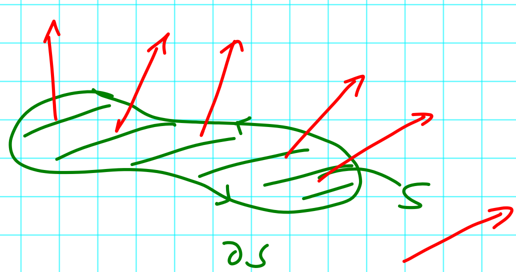
$\vec{j}$: elekt. Stromdichte

Ampère-Gesetz

2) Faradaysche Induktionsgesetz:
"zeitlich veränderliches Magnetfeld induziert ein elekt. Wirbelfeld"

$$\boxed{\operatorname{rot} \vec{E} = -\dot{\vec{B}}}$$

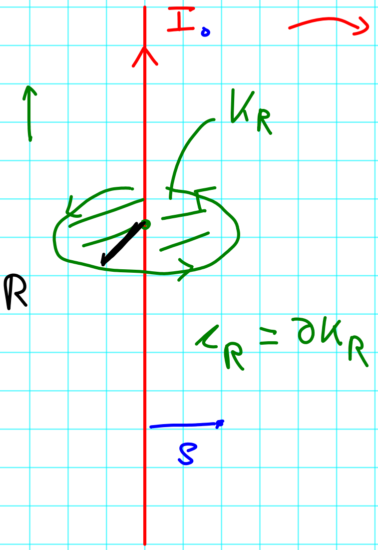
<u>Satz von Stokes</u>



$$\int_{\partial S} \vec{A}\, d\vec{l} = \int_S \operatorname{rot} \vec{A}\, d\vec{f}$$

Anwendungsbeispiel

$$\text{rot}\,\vec B = \mu_0\, \vec j_0$$



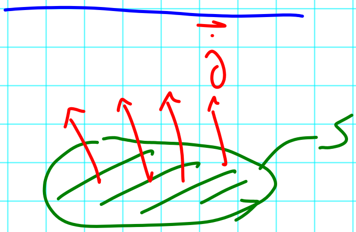
$$I_0 \to \vec j_0 \to \vec B(\vec r)$$

$$I_0 \overset{!}{=} I_{K_R} = \frac{1}{\mu_0}\int_{\lambda_R} \vec B\, d\vec l = \frac{1}{\mu_0}\, B(R)\, 2\pi R$$

Ansatz: $\vec B = B(\rho)\, \vec e_\varphi$

$$\rightsquigarrow \boxed{\vec B = \frac{I_0}{2\pi\mu_0}\,\frac{1}{\rho}\cdot \vec e_\varphi}$$

$$\vec j = \frac{1}{\mu_0}\,\text{rot}\,\vec B$$



$$I_S = \int_S \vec j\, d\vec f = \frac{1}{\mu_0}\int_S \text{rot}\,\vec B\, d\vec f$$

$$= \frac{1}{\mu_0}\int_{\partial S} \vec B\, d\vec l$$

Stokes!

$$\boxed{\int_{\partial S} \vec B\, d\vec l = \mu_0\, I_S}$$