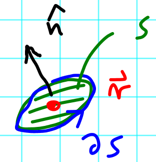
Wdhlg.: <u>Rotation</u> $\mathrm{rot}\,\vec{A}(\vec{r})$ = Vektor:

$$\boxed{\langle \hat{n}, \mathrm{rot}\,\vec{A}(\vec{r}) \rangle := \lim_{|S|\to 0} \frac{1}{|S|} \int_{\partial S} \vec{B}\, d\vec{\ell}}$$



$$\rightarrow \quad \boxed{\mathrm{rot}\,\vec{A} = \vec{\nabla} \times \vec{A}}$$

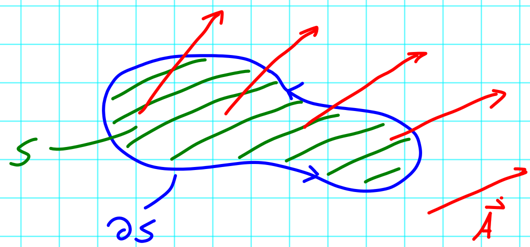
S: Fl.-Stück

$|S|$: Flächeninhalt

∂S: Flächenrand

$\hat{n}$: Flächennormale

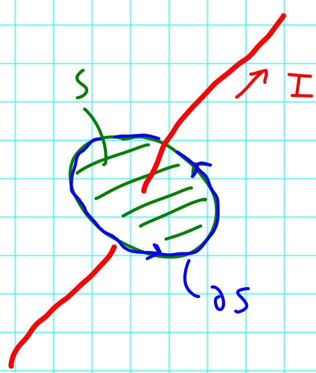
<u>Satz von Stokes</u>:



$$\boxed{\int_{\partial S} \vec{A}\, d\vec{\ell} = \int_{S} \mathrm{rot}\,\vec{A}\, d\vec{f}}$$

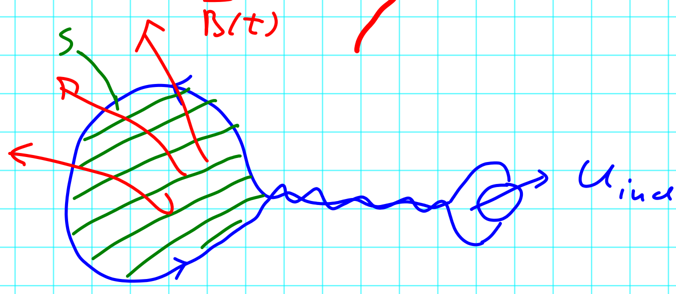
Magnetostatik: $\mathrm{rot}\,\vec{B} = \mu_0 \vec{j}$

$$\Rightarrow \quad \int_{\partial S} \vec{B}\, d\vec{\ell} = \mu_0 I$$



Faradaysche Induktion:

$$\mathrm{rot}\,\vec{E} = -\frac{\partial}{\partial t} \vec{B} \quad (*)$$



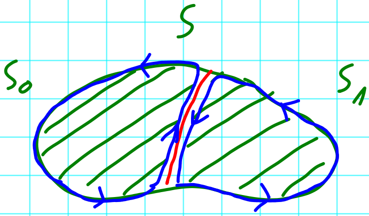
$$\Rightarrow \quad U_{ind} = -\frac{d}{dt} \Phi(t)$$

$$\int_{\partial S} \vec{E}\, d\vec{\ell} \underset{\text{Stokes}}{=} \int_S \mathrm{rot}\,\vec{E}\, d\vec{f} \underset{(*)}{=} -\frac{d}{dt} \int_S \vec{B}\, d\vec{f} \qquad \int_S \vec{B}(t)\, d\vec{f}$$

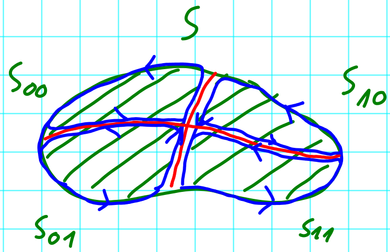
S. v. Stokes :

$$\int_{\partial S} \vec{A}\, d\vec{l} \overset{!}{=} \int_S \operatorname{rot} \vec{A}\, d\vec{f}$$

"Physikerbeweis":

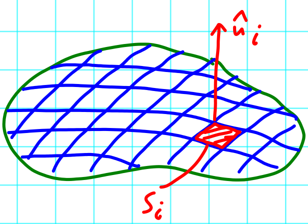


$$\int_{\partial S} \vec{A}\, d\vec{l} \overset{!}{=} \int_{\partial S_0} \vec{A}\, d\vec{l} + \int_{\partial S_1} \vec{A}\, d\vec{l}$$



$$\overset{!}{=} \int_{\partial S_{00}} \vec{A}\, d\vec{l} + \int_{\partial S_{01}} \vec{A}\, d\vec{l} + \int_{\partial S_{10}} \vec{A}\, d\vec{l} + \int_{\partial S_{11}} \vec{A}\, d\vec{l}$$

$$\vdots$$



$$= \sum_i \underbrace{\frac{1}{|S_i|} \int_{\partial S_i} \vec{A}\, d\vec{l}}_{\langle \hat{n}_i,\, \operatorname{rot} \vec{A}(\vec{r}_i) \rangle} \; |S_i| \qquad |S_i| \to 0$$

$$= \sum_i \langle \underbrace{\operatorname{rot} \vec{A}(\vec{r}_i)}, \underbrace{|S_i|\, \hat{n}_i} \rangle$$

$$= \int_S \operatorname{rot} \vec{A}\, d\vec{f}$$

Bemerkungen:

1) $\vec{A}$ konservativ $\Longrightarrow$ $\operatorname{rot} \vec{A} = \vec{0}$

$\ulcorner$ $\hookrightarrow$ $\vec{A} = -\operatorname{grad} V$ $\rightarrow$

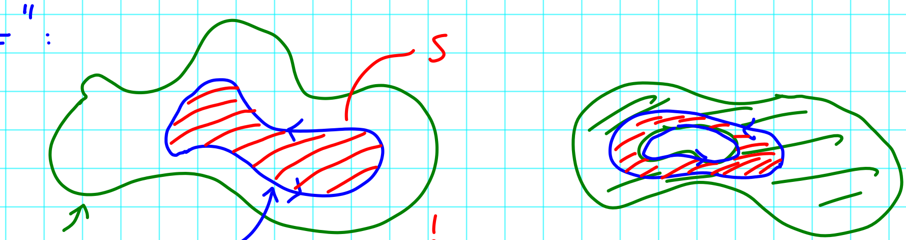
$$\operatorname{rot} \vec{A} = -\operatorname{rot} \operatorname{grad} V = -\underbrace{\vec{\nabla} \times \vec{\nabla}}_{= \vec{0}} V = \vec{0}$$

$$\vec{a} \times \vec{a} = \vec{0}$$

2) ist Vf. $\vec{A}$ auf einfach zusammenhängendem Gebiet G definiert, so gilt auch Umkehrung von 1):

$$\vec{A} \text{ konservativ} \iff \text{rot}\,\vec{A} = 0$$

„$\Leftarrow$":



G bel. geschl. Weg $c \overset{!}{=} \partial S$

$$\rightarrow \int_c \vec{A}\, d\vec{l} = \int_{\partial S} \vec{A}\, d\vec{l} \underset{\text{Stokes}}{=} \int_S \underbrace{\text{rot}\,\vec{A}}_{\overset{!}{=}0}\, d\vec{f} = 0$$

d.h. $\vec{A}$ integriert längs eines bel. geschlossenen Wegs ergibt 0 $\rightarrow$ $\vec{A}$ konservativ.

Laplace-Operator

$$\Delta u := \text{div grad}\, u = \vec{\nabla}\cdot\vec{\nabla} u$$

u: Skalarfeld

$$= \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}$$

d.h.
$$\boxed{\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}}$$

Anwendung: Elektrostatik

$$\rightarrow \quad \mathrm{rot}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} = \vec{0}$$

$\rightarrow$ $\vec{E}$ konservativ! $\rightarrow$ $\exists$ elekt. Potenzial U:

$$\vec{E} = -\,\mathrm{grad}\, U$$

$\vec{E}$ genügt $\quad \mathrm{div}\,\vec{E} = \rho / \varepsilon_0$

d.h. $$\underbrace{\mathrm{div}\ \overbrace{\mathrm{grad}\, U}^{\| \;\cdot(-1)}}_{\| \; \Delta} = -\rho/\varepsilon_0$$

$\rightarrow$ Poisson-Gleichung für elekt. Pot. U:

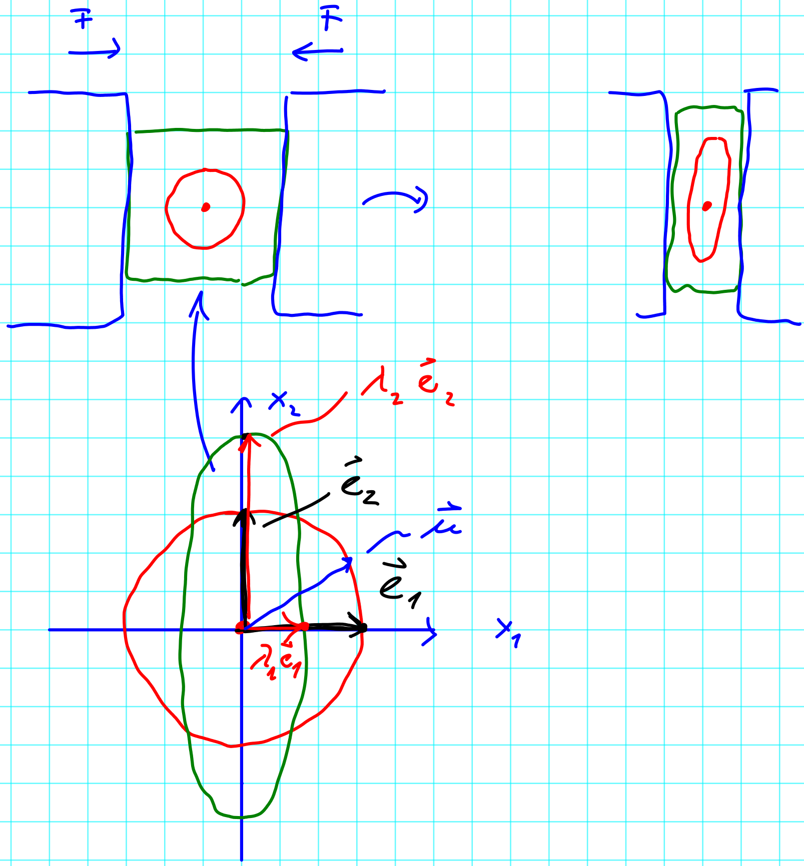
$$\boxed{\Delta U = -\rho / \varepsilon_0}$$

Neues Thema: Lineare Abbildungen, Matrizen, Eigenwerte, Eigenvektor, Eigenbasis

(→ Mechanik, QM)

Lineare Abbildungen

Beispiel: Deformation eines Festkörpers



→ beschreibe Deformation durch lineare Abb.

$A : \mathbb{R}^2 \to \mathbb{R}^2$:

bekannt:

$$A(\vec{e}_1) = \lambda_1 \vec{e}_1 \qquad (\lambda_1 < 1)$$

$$A(\vec{e}_2) = \lambda_2 \vec{e}_2 \qquad (\lambda_2 > 1)$$

$$A(\vec{u}) = \,?$$

→ nutze Linearität der Abb. aus!

Def.: Abb. $A: V \longrightarrow W$, wobei V und W VRe, ist <u>linear</u>

g.d.w.:

(i) $A(\vec{u} + \vec{v}) = A(\vec{u}) + A(\vec{v})$

(ii) $A(\lambda \vec{u}) = \lambda A(\vec{u})$

- Notation: $A(\vec{u}) = A\vec{u}$

- Begriffe: lineare Abb $\equiv$ linearer Operator

$\equiv$ Operator

$\equiv$ Matrix

Deformation:

$$A\vec{e}_1 = \lambda_1 \vec{e}_1$$
$$A\vec{e}_2 = \lambda_2 \vec{e}_2$$

allg. $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1 \vec{e}_1 + u_2 \vec{e}_2$

$$\rightarrow A\vec{u} = A(u_1 \vec{e}_1 + u_2 \vec{e}_2)$$

$$\overset{A \text{ linear}}{=} A(u_1 \vec{e}_1) + A(u_2 \vec{e}_2)$$

$$= u_1 \overbrace{A\vec{e}_1}^{\lambda_1 \vec{e}_1} + u_2 \overbrace{A\vec{e}_2}^{\lambda_2 \vec{e}_2} = \lambda_1 u_1 \vec{e}_1 + \lambda_2 u_2 \vec{e}_2$$

$$= \begin{pmatrix} \lambda_1 u_1 \\ \lambda_2 u_2 \end{pmatrix}$$

Verallgemeinerung $\longrightarrow$ <u>Abbildungsmatrix</u>

lineare Abb. $A: V \longrightarrow W$

$B = (\vec{e}_1, \dots, \vec{e}_n)$ Basis von V, $n = \dim V$

$C = (\vec{f}_1, \dots, \vec{f}_k)$ Basis von W, $k = \dim W$

betrachte Bilder der Basisvektoren $\vec{e}_1, \dots, \vec{e}_n$:

$$\underset{V}{\overset{\Uparrow}{A\,\vec{e}_1}} =: \underset{W}{\overset{\Uparrow}{\vec{a}_1}} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{l1} \end{pmatrix}_{C'}$$

$$A\,\vec{e}_2 =: \vec{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{l2} \end{pmatrix}_{C'}$$

$$\vdots$$

$$A\,\vec{e}_n =: \vec{a}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{ln} \end{pmatrix}_{C'}$$

Bild $A\vec{u}$ eines allg. Vektors $\vec{u} \in V$:

$$\hookrightarrow \quad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \sum_{i=1}^{n} u_i \cdot \vec{e}_i$$

$$A\vec{u} = A\Big(\sum_i u_i \vec{e}_i\Big) \underset{A \text{ linear}}{=} \sum_i u_i \underbrace{A\,\vec{e}_i}_{\vec{a}_i} = \sum_i u_i \vec{a}_i$$

im Komponenten bzgl. Basis C':

$$A\vec{u} = \sum_i u_i \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{li} \end{pmatrix}_{C'} = \begin{pmatrix} \sum_i a_{1i} u_i \\ \sum_i a_{2i} u_i \\ \vdots \\ \sum_i a_{li} u_i \end{pmatrix}_{C'}$$

$$=: \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{l1} & a_{l2} & \cdots & a_{ln} \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

Matrix – Vektor – Produkt

$$\begin{pmatrix} u_1 a_{11} \\ u_1 a_{21} \\ \vdots \\ u_1 a_{k1} \end{pmatrix} + \begin{pmatrix} u_2 a_{12} \\ u_2 a_{22} \\ \vdots \\ u_2 a_{k2} \end{pmatrix} + \cdots \begin{pmatrix} a_{1n} u_n \\ a_{2n} u_n \\ \vdots \\ a_{kn} u_n \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} u_1 a_{11} + a_{12} u_1 + \cdots + a_{1n} u_1 \\ u_2 a_{21} + a_{22} u_2 + \cdots + a_{2n} u_2 \\ \vdots \end{pmatrix}}_{!!} = \begin{pmatrix} \sum a_{1i} u_i \\ \sum a_{2i} u_i \\ \vdots \end{pmatrix}$$

$$k\left\{ \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & & a_{kn} \end{pmatrix}}_{n} \right. \cdot \left. \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \right\} n$$

Abbildungsmatrix von A bzgl. Basen B und C

$$_C A_B := (\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & & a_{kn} \end{pmatrix}$$

Bilder der Basisvektoren

= Spalten der Abbildungsmatrix !

Bsp lin. Abb $A : \mathbb{R}^3 \to \mathbb{R}^2$ $\quad B = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$

$C = (\vec{e}_1, \vec{e}_2)$

$\vec{a}_1 = A\vec{e}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $A\vec{e}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \vec{a}_2$

$\vec{a}_3 = A\vec{e}_3 = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

$\to$ Abb. matrix $_C A_B = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$

$$\vec{u} = \begin{pmatrix}2\\-2\\1\end{pmatrix} \qquad A\vec{u} = \vec{w} = {}_{\underline{\mathcal{C}}}A_{\mathcal{B}} \underbrace{\begin{pmatrix}2\\-2\\1\end{pmatrix}_{\mathcal{B}}}$$

$$= \begin{pmatrix}1 & 3 & 5\\2 & 4 & 6\end{pmatrix}\begin{pmatrix}2\\-2\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}_{\underline{\mathcal{C}}}$$

Deformation:

$$A\vec{e}_1 = \lambda_1\vec{e}_1 = \vec{a}_1 = \begin{pmatrix}\lambda_1\\0\end{pmatrix}$$

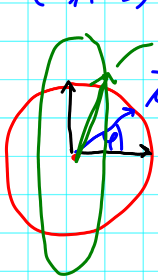
$$A\vec{e}_2 = \lambda_2\vec{e}_2 = \vec{a}_2 = \begin{pmatrix}0\\\lambda_2\end{pmatrix}$$

$A: \mathbb{R}^2 \to \mathbb{R}^2$

$\mathcal{B} = (\vec{e}_1, \vec{e}_2)$

$\mathcal{B} = \mathcal{C} = (\vec{e}_1, \vec{e}_2)$

$$\Rightarrow \quad {}_{\mathcal{B}}A_{\mathcal{B}} = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}$$



$$\vec{u}_\varphi = \begin{pmatrix}\cos\varphi\\\sin\varphi\end{pmatrix}$$

$$\vec{w} = A\vec{u}_\varphi = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}\begin{pmatrix}\cos\varphi\\\sin\varphi\end{pmatrix} = \begin{pmatrix}\lambda_1\cos\varphi\\\lambda_2\sin\varphi\end{pmatrix}$$

$$\vec{w} = \begin{pmatrix}\lambda_1\cos\varphi\\\lambda_2\sin\varphi\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$$

$$\left(\frac{x}{\lambda_1}\right)^2 + \left(\frac{y}{\lambda_2}\right)^2 = 1$$