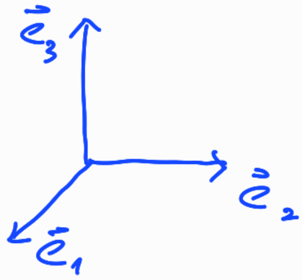
letzte Woche:

- VR mit Skalarprodukt $\rightsquigarrow$ Geometrie

- **Orthonormalbasis**

$B = (\vec{e}_1, \dots, \vec{e}_n)$ ONB g.d.w. $\vec{e}_i$ normiert und paarweise orthogonal

d.h. $$\boxed{\langle \vec{e}_i, \vec{e}_j \rangle = \delta_{ij}}$$



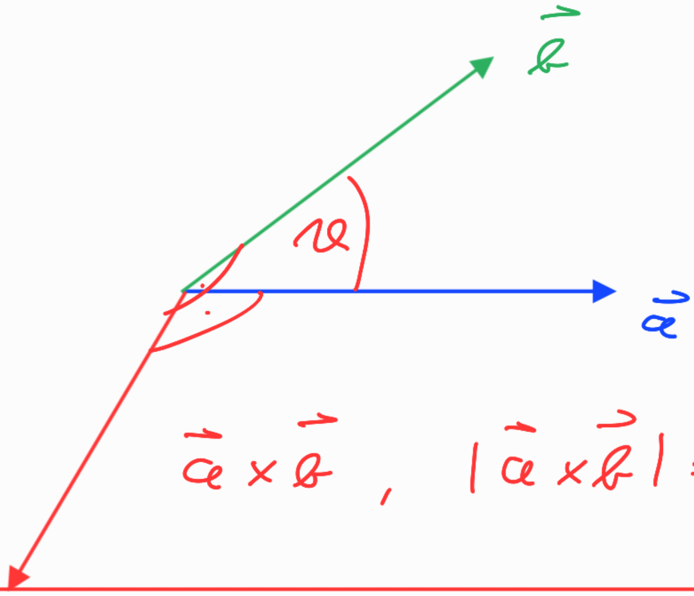
$\rightsquigarrow$ • $\langle \vec{a}, \vec{b} \rangle = \sum_{i=1}^{n} a_i b_i$

• $a_i = \langle \vec{e}_i, \vec{a} \rangle$

- **Induktionsbeweis**

Vektorprodukt (= Kreuzprodukt) für dreidimensionalen eukl. VR

geometrisch:



$\vec{a} \times \vec{b}$, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \cdot |\sin\vartheta|$

Def.: Vektorprodukt $\equiv$ Abb. $V \times V \longrightarrow V$
(dim $V = 3$) $\quad \vec{a}, \vec{b} \longmapsto \vec{a} \times \vec{b}$

mit Eigenschaften

(V1) $\quad \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (Antisymmetrie)

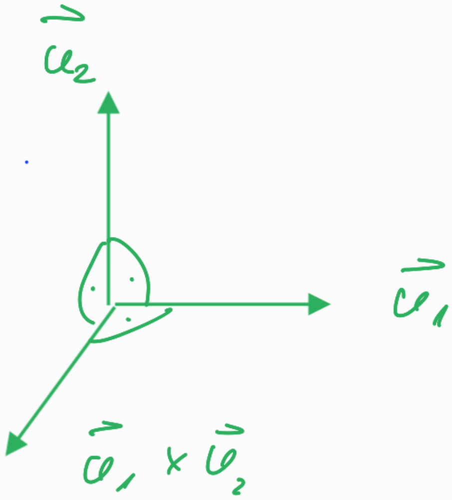
(V2) $\quad \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$\quad \vec{a} \times (\lambda \vec{b}) = \lambda\, \vec{a} \times \vec{b}$

(V3) $\quad \vec{u}_1, \vec{u}_2$ orthonormal

$\Rightarrow \vec{u}_1, \vec{u}_2, \vec{u}_1 \times \vec{u}_2$

rechtshändige ONB



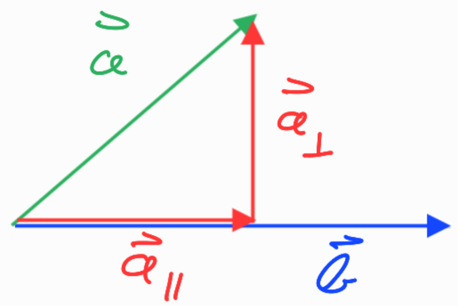
→ weitere Eigenschaften und Anwendungen:

1) $\vec{a} \times \vec{a} = \vec{0}$

2) $\vec{a} \parallel \vec{b} \iff \vec{a} \times \vec{b} \neq \vec{0}$

3) <u>Orthogonalkomponente</u>

$$\left(\vec{a}_{\parallel} = \langle \hat{b}, \vec{a} \rangle \, \hat{b} \right)$$



$$\boxed{\vec{a}_{\perp} = (\hat{b} \times \vec{a}) \times \hat{b}}$$

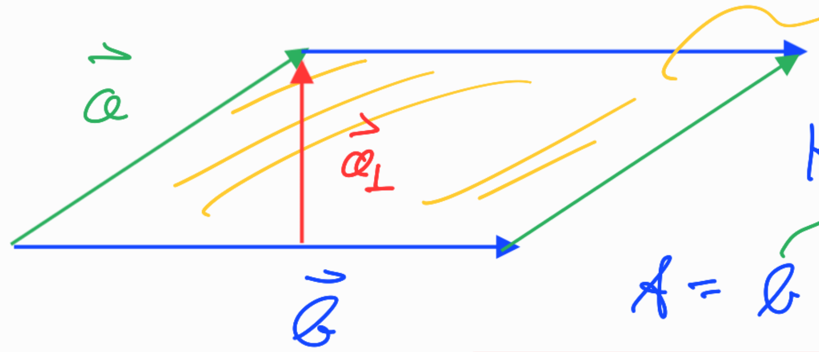
$$a_{\perp} = |\hat{b} \times \vec{a}|$$

$$\vec{b} \rightsquigarrow \hat{b} := \frac{1}{|\vec{b}|} \vec{b}$$

4) $$|\vec{a} \times \vec{b}| = |\vec{a}| \, |\vec{b}| \, |\sin\alpha|$$

5) Flächeninhalt eines Parallelogramms

$$A = b \cdot H$$

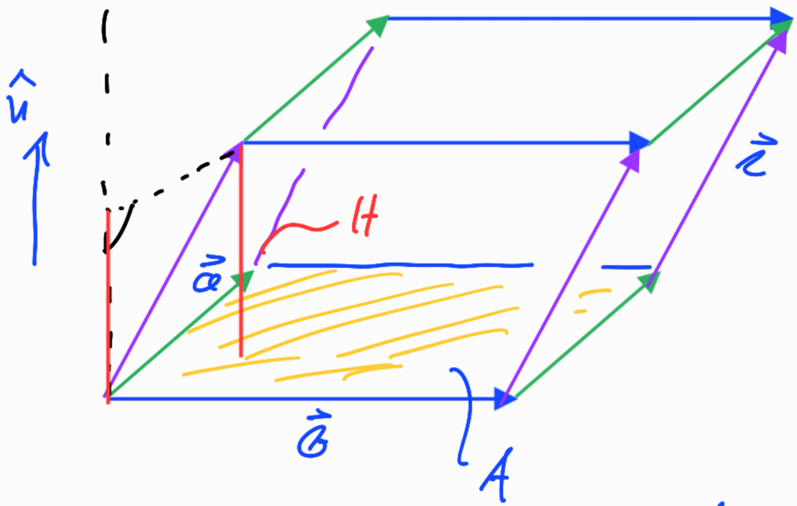


$$H = a_{\perp} = |\hat{b} \times \vec{a}|$$

$$A = b \, |\hat{b} \times \vec{a}| = |\vec{b} \times \vec{a}|$$

$$\boxed{A = |\vec{a} \times \vec{b}|}$$

6) Volumeninhalt eines Spats (Parallelepiped)



$V = A \cdot H$

$A = |\vec{a} \times \vec{b}|$ (1)

(2) $H = \langle \hat{n}, \vec{c} \rangle$, $\hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ (3)

$$\rightsquigarrow V = A \cdot H \overset{1),2)}{=} |\vec{a} \times \vec{b}| \langle \hat{n}, \vec{c} \rangle$$

$$\overset{(3)}{=} \cancel{|\vec{a} \times \vec{b}|} \langle \frac{\vec{a} \times \vec{b}}{\cancel{|\vec{a} \times \vec{b}|}}, \vec{c} \rangle$$

$$\boxed{V = |\langle \vec{a} \times \vec{b}, \vec{c} \rangle|}$$

Spatprodukt

7) Berechnung in Komp. bzgl. ONB B:

$$\vec{a} \times \vec{b} = \begin{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_B \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \end{pmatrix}_B = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}_B$$

Begründungen von 1) – 7):

zu 1) $\vec a \times \vec a = -\vec a \times \vec a \rightsquigarrow \vec a \times \vec a = \vec 0$

zu 2) $\vec a \parallel \vec b \Leftrightarrow \vec a \times \vec b = \vec 0$

„$\Rightarrow$" ↳ $\vec b = \lambda \vec a \rightarrow \vec a \times \vec b = \vec a \times (\lambda \vec a) = \lambda (\vec a \times \vec a) = \vec 0$

„$\Leftarrow$" z.z.: $\vec a \nparallel \vec b \Rightarrow \vec a \times \vec b \neq \vec 0$

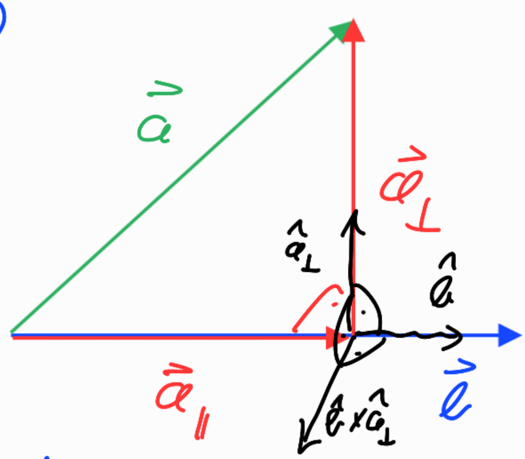
↳ $\vec a = \vec a_\parallel + \vec a_\perp \rightsquigarrow \vec a \times \vec b = (\vec a_\parallel + \vec a_\perp) \times \vec b$

$a_\perp \neq 0$

$$= \underbrace{\vec a_\parallel \times \vec b}_{=\vec 0} + \vec a_\perp \times \vec b =$$

$$= (|\vec a_\perp| \hat a_\perp) \times (|\vec b| \hat b) = \underbrace{|\vec a_\perp|}_{\neq 0} \underbrace{|\vec b|}_{\neq 0} \underbrace{\hat a_\perp \times \hat b}_{\neq \vec 0 \ (V3)} \neq \vec 0$$

zu 3)



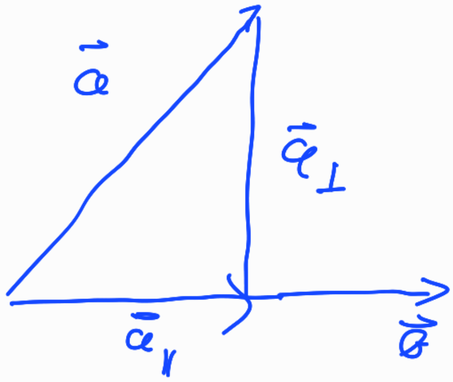
$\vec a = \vec a_\parallel + \vec a_\perp$

$$(\hat b \times \vec a) \times \hat b = (\overbrace{\hat b \times (\vec a_\parallel}^{\rightsquigarrow \vec 0} + \vec a_\perp)) \times \hat b$$

$$= (\hat b \times \vec a_\perp) \times \hat b = |\vec a_\perp| \underbrace{(\hat b \times \hat a_\perp) \times \hat b}_{= \hat a_\perp} = \vec a_\perp$$

$$\rightarrow \quad |\vec{a}_\perp| = |\hat{b} \times \vec{a}|$$

4)



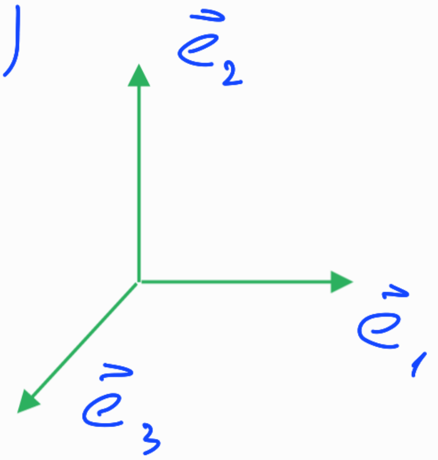
2.2: $|\vec{a} \times \vec{b}| \overset{!}{=} ab\,|\sin\vartheta|$

$$a_\perp^2 = a^2 - \underset{\left(\langle \hat{b}, \vec{a}\rangle\right)^2 = a^2 \cdot \cos^2\vartheta}{\underset{\shortparallel}{a_\parallel^2}} = a^2(1 - \cos^2\vartheta) = a^2 \sin^2\vartheta$$

$$\underset{\underset{a|\sin\vartheta|}{\shortparallel}}{b\, a_\perp} \overset{3)}{=} b\,|\hat{b} \times \vec{a}| = |\vec{b} \times \vec{a}| = ab\,|\sin\vartheta|$$

5) 6) $\checkmark$

zu 7) ONB $B = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$



rechtshändige ONB

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_B = \sum_{i=1}^{3} a_i \vec{e}_i$$

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_B = \sum_{j=1}^{3} b_j \vec{e}_j$$

$$\vec{a} \times \vec{b} = \sum_{i,j=1}^{3} (a_i \vec{e}_i) \times (b_j \vec{e}_j) = \sum_{i,j=1}^{3} a_i b_j \vec{e}_i \times \vec{e}_j$$

$$= \sum_{\substack{i,j=1 \\ i \neq j}}^{3} a_i b_j \vec{e}_i \times \vec{e}_j = \sum_{\substack{i,j=1 \\ i<j}}^{3} \left(a_i b_j \vec{e}_i \times \vec{e}_j + a_j b_i \underbrace{\vec{e}_j \times \vec{e}_i}_{= -\vec{e}_i \times \vec{e}_j} \right)$$

$$= \sum_{i<j} (a_i b_j - a_j b_i) \vec{e}_i \times \vec{e}_j =$$

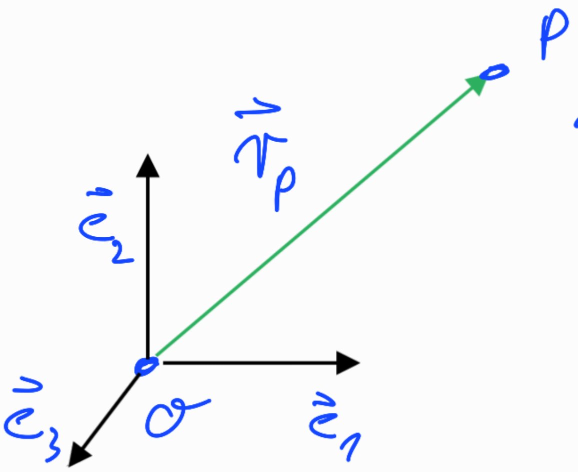
$$\rightarrow \quad \underline{\vec a \times \vec b} = (a_1 b_2 - a_2 b_1)\overbrace{(\vec e_1 \times \vec e_2)}^{= \vec e_3}$$
$$+ (a_1 b_3 - a_3 b_1)\overbrace{(\vec e_1 \times \vec e_3)}^{= -\vec e_2}$$
$$+ (a_2 b_3 - a_3 b_2)\underbrace{(\vec e_2 \times \vec e_3)}_{= +\vec e_1}$$

$$= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}_B \quad \checkmark$$

Koordinatensysteme für den 3D Raum

2) kartesisches Koordinatensystem:

- <u>Bezugspunkt (Ursprung) $\mathcal{O}$</u>



$\vec r_P$ = <u>Ortsvektor des Punkts P</u>

<u>bzgl. $\mathcal{O}$</u>:

- <u>ONB $B = (\vec e_1, \vec e_2, \vec e_3)$</u> $\rightsquigarrow \quad \vec r_P = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_B$

x_1, x_2, x_3 : kartesische Koordinaten (bzgl. Bezugspunkt $\mathcal{O}$ und ONB B)

Koordinatenlinien: $(x_3 = 0)$