Wdhlg.:

Taylor-Entwicklung n-ter Ord. von $f(x)$ um $x=0$

$$\boxed{f(x) = \underbrace{\sum_{m=0}^{n} \frac{f^{(m)}(0)}{m!} x^m}_{=: \tilde{f}_n(x)} + o(x^{n+1})}$$

↳ Potenzreihe / Taylor-Reihe einer analyt. Fkt. $f(x)$ um $x=0$:

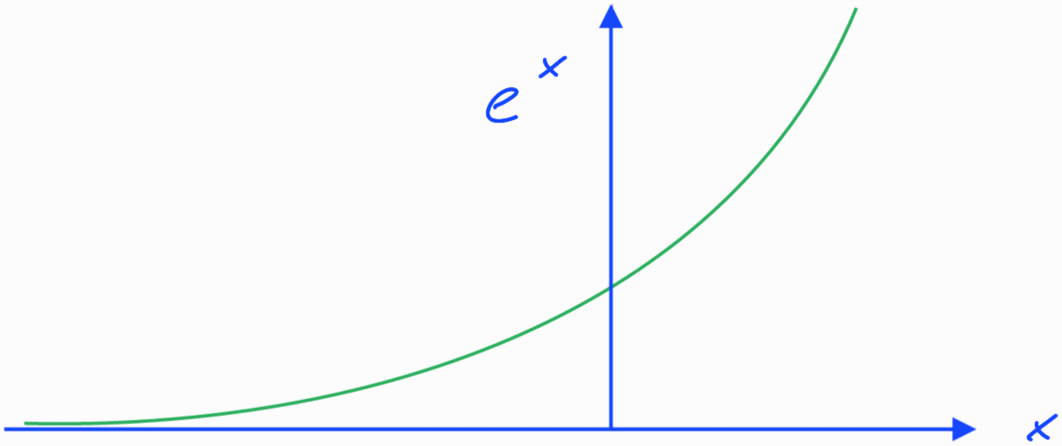
$$\boxed{f(x) = \sum_{m=0}^{\infty} a_m x^m}$$

↳ $a_m = \dfrac{f^{(m)}(0)}{m!}$

Exponentialfunktion:

$$\boxed{e^x = \exp(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!}}$$

Euler-Zahl $e = \sum_{m=0}^{\infty} \frac{1}{m!} = 2{,}718\ldots \quad \left(\stackrel{!}{=} e^1\right)$



$$\boxed{(e^x)' = e^x}$$

- $e^x e^y = e^{x+y}$
- $(e^x)^\lambda = e^{\lambda x}$

<u>(Natürliche) Logarithmus</u>

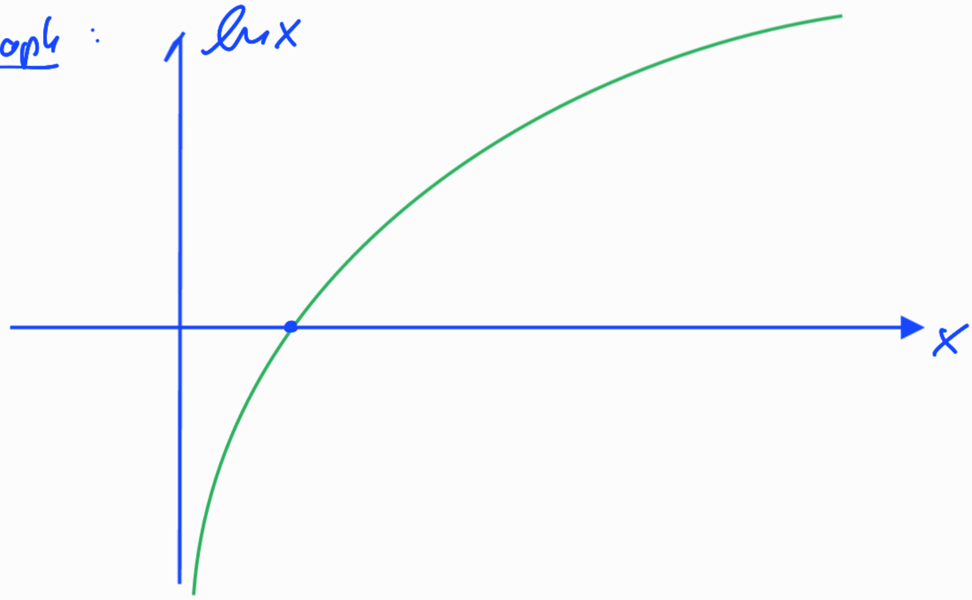
$$\ln := \exp^{-1} : \mathbb{R}_+ \to \mathbb{R}$$
$$x \mapsto \ln x$$

d.h. $\ln \circ \exp = id_{\mathbb{R}}$

$\exp \circ \ln = id_{\mathbb{R}_+}$

$\ln e^x = x \quad ; \quad e^{\ln x} = x$

Funktionsgraph:



Ableitung:

$$e^{\ln x} = x \qquad \Big| \frac{d}{dx}$$

$$\underbrace{e^{\ln x}}_{x} \cdot (\ln x)' = 1$$

d.h. $$\boxed{(\ln x)' = \frac{1}{x}}$$

Rechenregeln:

- $\ln(ab) = \ln a + \ln b$
- $\ln(a^\lambda) = \lambda \ln a$
- $\ln(a/b) = \ln a - \ln b$

$$\Big[\ \ln(ab) = \ln\left(e^{\ln a} e^{\ln b}\right) = \ln\left(e^{\ln a + \ln b}\right)$$

$$= \ln a + \ln b$$

$$\ln(a^\lambda) = \ln\left(\left(e^{\ln a}\right)^\lambda\right) = \ln\left(e^{\lambda \ln a}\right) = \lambda \ln a$$

$$\ln(a/b) = \ln\left(a \cdot \tfrac{1}{b}\right) = \ln a + \ln\left(b^{-1}\right) = \ln - \ln b\ \Big]$$

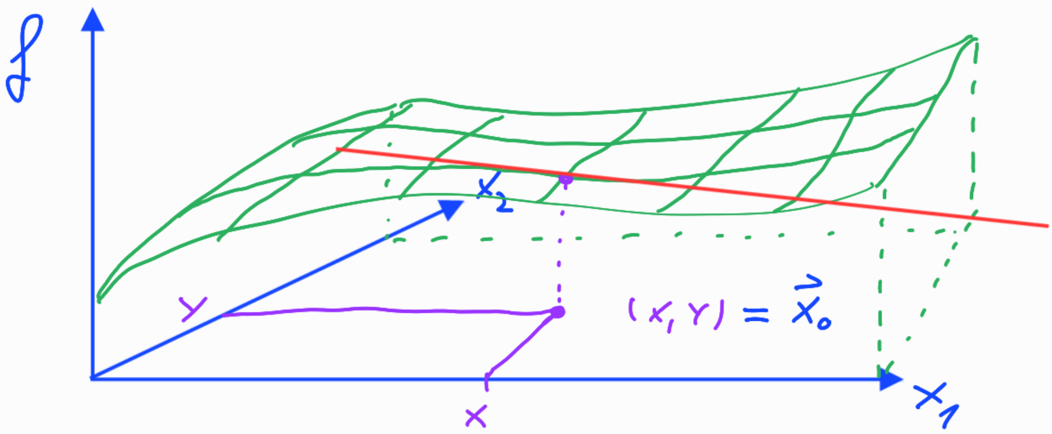
Partielle Ableitung und Gradient

einer Fkt. $f : D \to \mathbb{R} \qquad D \subset \mathbb{R}^n$

$\vec{x} = (x_1, x_2, \dots x_n) \mapsto f(x_1, x_2, \dots, x_n)$

$= f(\vec{x})$

$\underline{\underline{n=2}}$: Funktionsgraph:



$\frac{df}{dx}$

Steigung der Tangente bei $\vec{x}_0$ in x_1 - Richtung = "partielle Ableitung von f in $\vec{x}_0$ nach x_1":

$$\frac{\partial f}{\partial x_1}(\vec{x}_0) = \left(f(x, y) \right)' = \lim_{h \to 0} \frac{1}{h} \left(f(x+h, y) - f(x, y) \right)$$

$\vec{x}_0 = (x, y)$

$$\frac{\partial f}{\partial x_2}(\vec{x}_0) = \left(f(x, y) \right)' = \lim_{h \to 0} \frac{1}{h} \left(f(x, y+h) - f(x, y) \right)$$

"∂" = "del" ($\neq \delta$ = "delta")

Def: Partielle Ableitung von $f: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x_i}(\vec{x}) &= \Big(f(x_1, x_2, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) \Big)' \\ &= \lim_{h \to 0} \frac{1}{h} \Big(f(x_1, \dots, x_i + h, \dots) - f(\vec{x}) \Big) \\ &= \lim_{h \to 0} \frac{1}{h} \Big(f(\vec{x} + h\,\vec{e}_i) - f(\vec{x}) \Big) \end{aligned}}$$

Bsp: $f(x_1, x_2) = x_1^2 \cos x_2$

$$\hookrightarrow \frac{\partial f}{\partial x_1}(\vec{x}) = 2 x_1 \cos x_2$$

$$\frac{\partial f}{\partial x_2}(\vec{x}) = -x_1^2 \sin x_2$$

Lineare Näherung:

$$f(\vec{x} + \vec{h}) = f(x_1 + h_1, x_2 + h_2, \dots, x_n + h_n)$$

$|\vec{h}| \ll 1$

$$= f(x_1, x_2 + h_2, x_3 + h_3, \dots, x_n + h_n) + \frac{\partial f}{\partial x_1}(\vec{x})\, h_1$$

$$= f(x_1, x_2, x_3 + h_3, \dots, x_n + h_n) + \frac{\partial f}{\partial x_1}(\vec{x})\, h_1 + \frac{\partial f}{\partial x_2}(\vec{x})\, h_2$$

$$\vdots$$

$$= f(\vec{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}) \cdot h_i$$

$$f(\vec{x}+\vec{h}) = f(\vec{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}) \cdot h_i + O(|\vec{h}|^2)$$

$$\langle \vec{a}, \vec{h} \rangle = \sum_i a_i h_i$$

$$\boxed{\begin{array}{l} \underline{\textbf{Def.}}: \ \underline{\text{Gradient der Fkt. } f(\vec{x}) \text{ in } \vec{x}_0} \\ \operatorname{grad} f(\vec{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x})\, \vec{e}_i = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\vec{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\vec{x}) \end{pmatrix} \end{array}}$$

Nabla-Notation:

$$\vec{\nabla} = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \vdots \\ \partial/\partial x_n \end{pmatrix} ; \quad \vec{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

$$\operatorname{grad} f(\vec{x}) = \vec{\nabla} f(\vec{x})$$

$$\Rightarrow \boxed{f(\vec{x}+\vec{h}) = f(\vec{x}) + \langle \operatorname{grad} f(\vec{x}), \vec{h} \rangle} + O(|\vec{h}|^2)$$

$$\frac{25}{17} \cdot |\vec{h}|^2$$

$\boxed{\begin{array}{l}\underline{\text{Def.}}: \underline{\text{Gradient der Fkt. } f(\vec{x}) \text{ in } \vec{x}_0} \\ \operatorname{grad} f(\vec{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x})\, \vec{e}_i = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\vec{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\vec{x}) \end{pmatrix}\end{array}}$

$$\boxed{f(\vec{x}+\vec{h}) = f(\vec{x}) + \langle \operatorname{grad} f(\vec{x}), \vec{h} \rangle}$$

<u>Bsp.</u>: $f(\vec{x}) = (x_1+1)^2 (x_2+1) \cos x_3$

Lin. Näherung in $\vec{x} = \vec{0}$:

$$\operatorname{grad} f(\vec{0}) = \left.\begin{pmatrix} 2(x_1+1)(x_2+1)\cos x_3 \\ (x_1+1)^2 \cos x_3 \\ -(x_1+1)^2 (x_2+1) \sin x_3 \end{pmatrix}\right|_{\vec{x}=\vec{0}} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\rightarrow \quad f(\vec{h}) = f(\vec{0}) + \langle \operatorname{grad} f(\vec{0}), \vec{h} \rangle$$

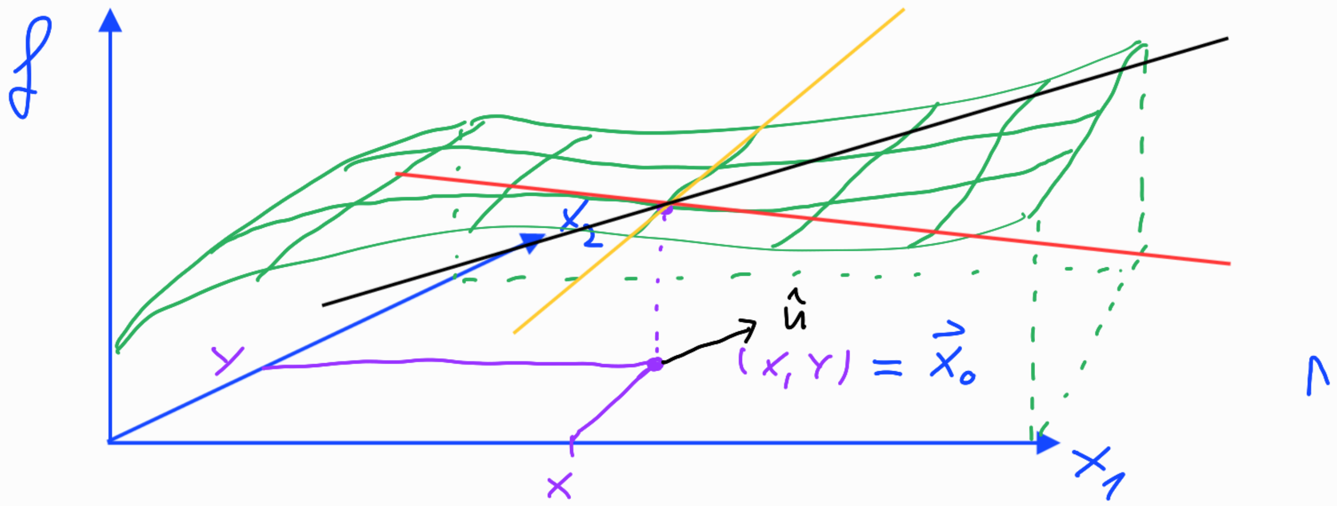
$$= 1 + \left\langle \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \right\rangle$$

$$f(h_1, h_2, h_3) = 1 + 2h_1 + h_2$$

$\ulcorner\, n \geq 1: \quad n! := n\cdot(n-1)\cdots 3\cdot 2\cdot 1$

$0! := 1$ $\lrcorner$

Richtungsableitung von f in $\vec{x}_0$ in Richtung $\hat{u}$



$$\boxed{\partial_{\hat{u}} f(\vec{x}) := \lim_{h\to 0} \frac{1}{h}\Big(f(\vec{x} + h\,\hat{u}) - f(\vec{x})\Big)}$$

$$\ulcorner\ \frac{\partial f}{\partial x_i}(\vec{x}) = \lim_{h \to 0} \frac{1}{h}\Big(f(\vec{x} + h\,\vec{e}_i) - f(\vec{x})\Big)$$

$$\shortparallel \qquad = \partial_{\vec{e}_i} f(\vec{x}) \quad \lrcorner$$

$$\big(\big(\ \partial_{x_i} f\ \big)\big)$$

$$\boxed{\begin{aligned}\partial_{\hat{u}} f(\vec{x}) &:= \lim_{h\to 0} \frac{1}{h}\Big(f(\vec{x}+h\hat{u}) - f(\vec{x})\Big) \\ &\overset{!}{=} \langle \operatorname{grad} f(\vec{x}), \hat{u} \rangle\end{aligned}} \quad (*) \qquad |\hat{u}| = 1$$

$$\ulcorner \quad \frac{1}{\cancel{h}}\Big(\underbrace{f(\vec{x}+h\hat{u})}_{\cancel{f(\vec{x})} + \langle \operatorname{grad} f(\vec{x}), \cancel{h}\hat{u}\rangle} - \cancel{f(\vec{x})} \Big) = \langle \operatorname{grad} f(\vec{x}), \hat{u} \rangle \quad \lrcorner$$

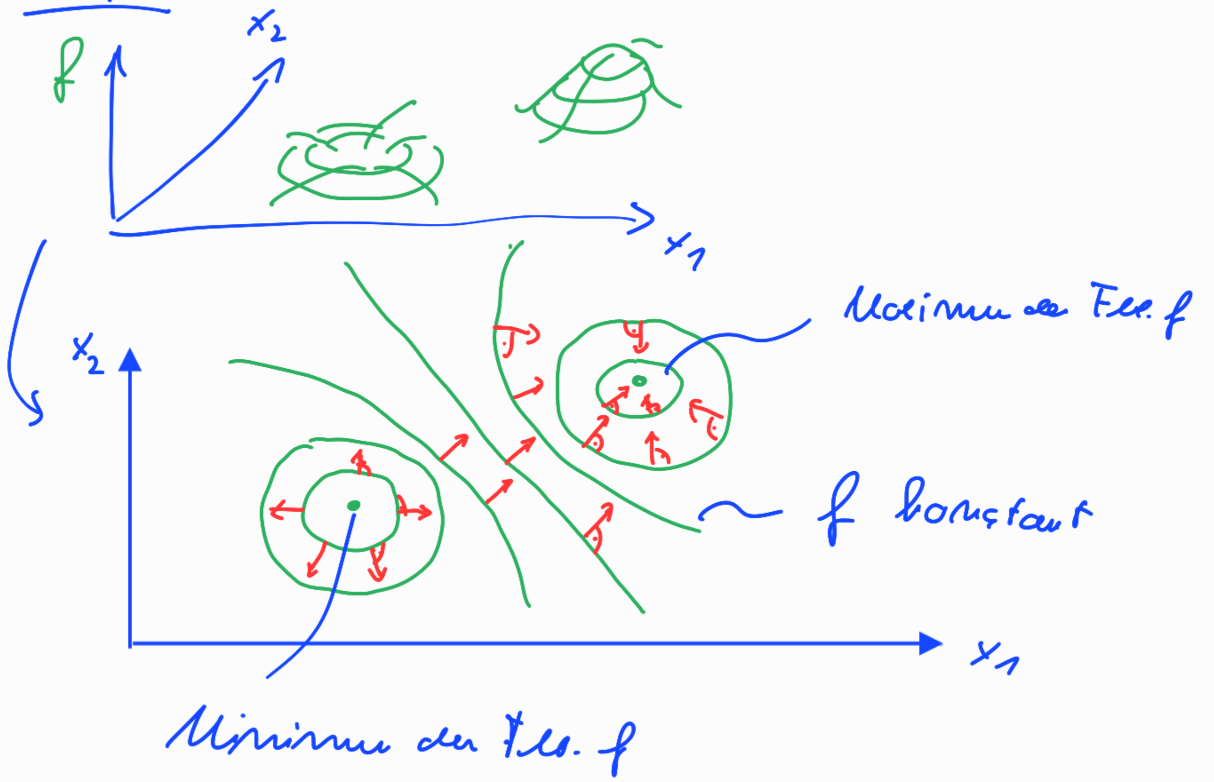
2> Eigenschaften des Gradienten

1) $\overbrace{\operatorname{grad} f(\vec{x})}$ = Richtung des stärksten Anstiegs von f in $\vec{x}$

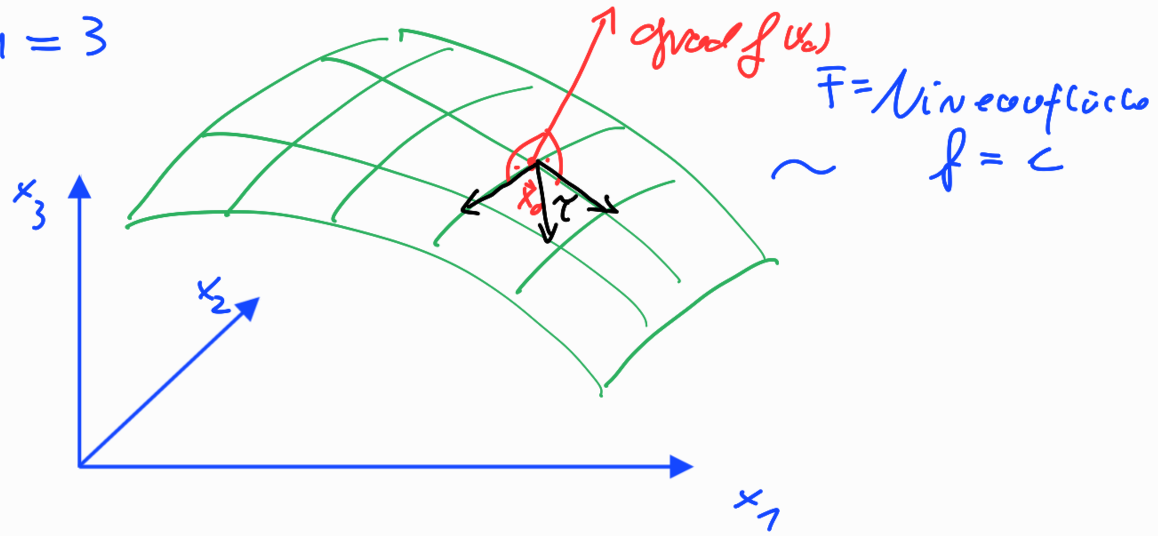
2) $|\operatorname{grad} f(\vec{x})|$ = maximale Steigung in $\vec{x}$

3) $\operatorname{grad} f(\vec{x}) \perp$ Niveauflächen von f

Beispiele: $n = 2$



$n = 3$



$$F = \{ \vec{x} \in \mathbb{R}^3 \mid f(\vec{x}) \overset{!}{=} c \}$$

$$\boxed{\vec{\tau} \text{ tangential zu } F \text{ in } \vec{x}_0 \quad :\Longleftrightarrow \quad \partial_{\hat{\tau}} f(\vec{x}_0) \overset{!}{=} 0}$$

$\vec{\tau}$ tangential $\leadsto$ $0 \overset{!}{=} \partial_{\hat{\tau}} f(\vec{x}_0) = \overset{(*)}{=} \langle \operatorname{grad} f(\vec{x}_0), \hat{\tau} \rangle$

d.h. $\operatorname{grad} f(\vec{x}_0) \perp \hat{\tau}$.