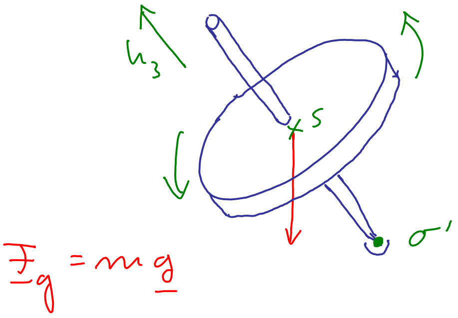
Dynamik des symmetrischen Kreisels:

Σ = axialsymmetrischer Körper (genauer: $I_1 = I_2 \neq I_3$)

frei drehbar fixiert in O', unter Drehmoment

$$\underline{M} = m g l \, \underline{h}_3 \times \underline{e}_3$$

etwa:



zuerst: $\underline{M} = 0$ (z.B. durch $O' = S$) $\rightarrow$ freie Rotation

Euler-Gleichungen:

$$\left.\begin{aligned} I_1 \dot{\omega}_1 + (I_3 - I_1)\,\omega_2 \omega_3 &= 0 \\ I_1 \dot{\omega}_2 + (I_1 - I_3)\,\omega_1 \omega_3 &= 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} \dot{\omega}_1 + \lambda \omega_2 &= 0 \\ \dot{\omega}_2 - \lambda \omega_1 &= 0 \end{aligned}\right\} (*)$$

$$I \dot{\omega}_3 = 0 \quad \rightarrow \omega_3 = \Omega$$

wobei

$$\lambda = \frac{I_3 - I_1}{I_1} \Omega$$

$$(*) \rightarrow \quad \ddot{\omega}_1 = -\lambda^2 \omega_1$$

d.h.

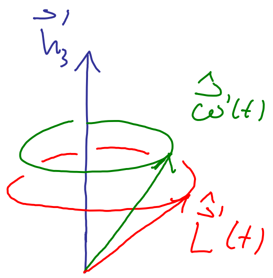
$$\omega_1(t) = a \cos(\lambda t + \cancel{\varphi})$$

$$\omega_2(t) = -\frac{1}{\lambda} \dot{\omega}_1(t) = a \sin(\lambda t + \cancel{\varphi})$$

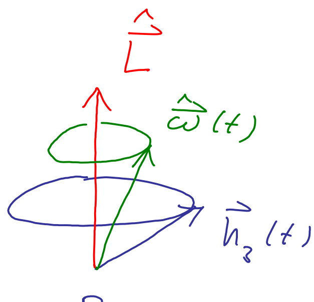
also

$$\vec{\omega}\,'(t) = \begin{pmatrix} a \cos \lambda t \\ a \sin \lambda t \\ \Omega \end{pmatrix}$$

$$\rightarrow \quad \vec{L}\,'(t) = \begin{pmatrix} I_1 a \cos \lambda t \\ I_1 a \sin \lambda t \\ I_3 \Omega \end{pmatrix}$$



B_0'

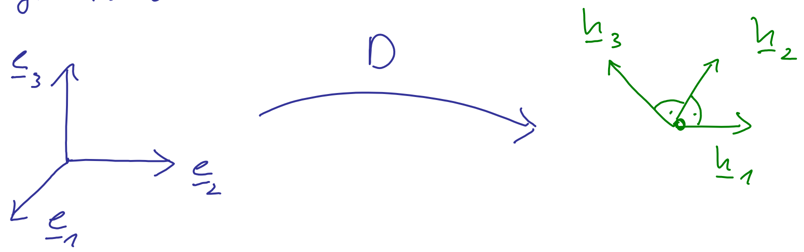


B

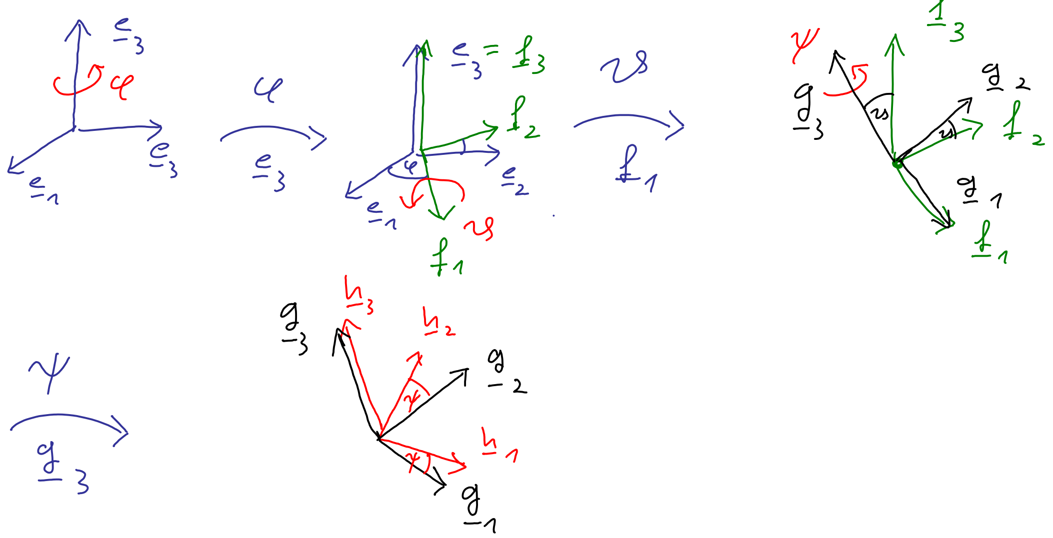
reguläre Präzession der Hauptachse h_3 um $\underline{L}$ mit Frequenz $|\lambda| = \left| \frac{I_3 - I_1}{I} \Omega \right|$.

Dynamik unter Moment $\underline{M} = mgl\, \underline{h}_3 \times \underline{e}_3$ behandeln wir mittels <u>Lagrangescher Mechanik</u>:

<u>Koordinaten</u>: <u>Euler-Winkel</u> φ, ϑ, ψ parametrisieren allg. Rotation



gemäß



$\rightarrow$ Potenzielle Energie: $U(\vartheta) = mgl \cos\vartheta$

Kinetische Energie: $T = \frac{1}{2} \vec{\omega}'^T \underline{I}\, \vec{\omega}'$

mit $\vec{\omega}'$ bestimmt durch

$$\underline{\omega} \overset{!}{=} \dot{\varphi}\, \underline{e}_3 + \dot{\vartheta}\, \underline{f}_1 + \dot{\psi}\, \underline{h}_3$$

und
$$\underline{f}_1 = \cos\psi\, \underline{h}_1 - \sin\psi\, \underline{h}_2$$
$$\underline{e}_3 = \cos\vartheta\, \underline{h}_3 + \sin\vartheta\, (\sin\psi\, \underline{h}_1 + \cos\psi\, \underline{h}_2),$$

$$\rightarrow \left.\begin{aligned} \omega_1' &= \dot{\varphi} \sin\vartheta \sin\psi + \dot{\vartheta} \cos\psi \\ \omega_2' &= \dot{\varphi} \sin\vartheta \cos\psi - \dot{\vartheta} \sin\psi \\ \omega_3' &= \dot{\varphi} \cos\vartheta + \dot{\psi} \end{aligned}\right\} (*)$$

d.h. $L = T - U = \frac{I_1}{2}\left(\omega_1'^2 + \omega_2'^2\right) + \frac{I_3}{2}\omega_3'^2 - mgl\cos\vartheta$

(*) $\rightarrow$ $$\boxed{L = \frac{I_1}{2}\left(\dot\varphi^2 \sin^2\vartheta + \dot\vartheta^2\right) + \frac{I_3}{2}\left(\dot\varphi\cos\vartheta + \dot\psi\right)^2 - mgl\cos\vartheta}$$

$\frac{\partial L}{\partial \varphi} = 0 \rightarrow P_\varphi = \frac{\partial L}{\partial \dot\varphi}$ konstant $(\hat{=} L_3)$

$\frac{\partial L}{\partial \psi} = 0 \rightarrow P_\psi = \frac{\partial L}{\partial \dot\psi}$ konstant $(\hat{=} L_3')$

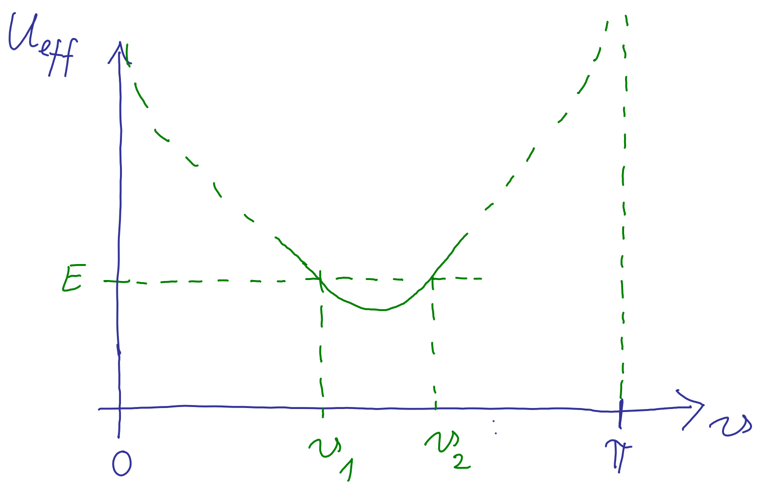
$P_\varphi = I_1 \dot\varphi \sin^2\vartheta + I_3(\dot\varphi\cos\vartheta + \dot\psi)\cos\vartheta = I_1\dot\varphi\sin^2\vartheta + P_\psi\cos\vartheta$

$P_\psi = I_3(\dot\varphi\cos\vartheta + \dot\psi)$

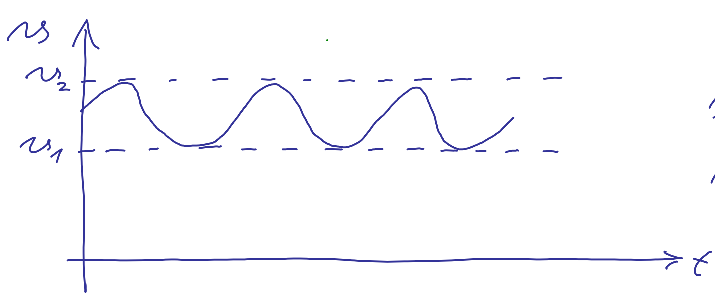
Erhaltung der Energie T+U lautet damit

$$E = \frac{I_1}{2}\dot\vartheta^2 + \underbrace{\frac{(P_\varphi - P_\psi\cos\vartheta)^2}{2I_1\sin^2\vartheta} + \cancel{\frac{P_\psi^2}{2I_3}} + mgl\cos\vartheta}_{=:\,U_{eff}(\vartheta)}$$

konstant!



$\rightarrow$ $\vartheta(t)$ oszilliert zwischen ϑ_1 und ϑ_2 !



Nutation der Achse $\underline{k}_3$

$$\dot{\varphi}(t) = \left(p_\varphi - p_\psi \cos\vartheta(t)\right) / I_1 \sin^2\vartheta(t)$$

Präzession der Achse $\underline{n}_3$

etwa: