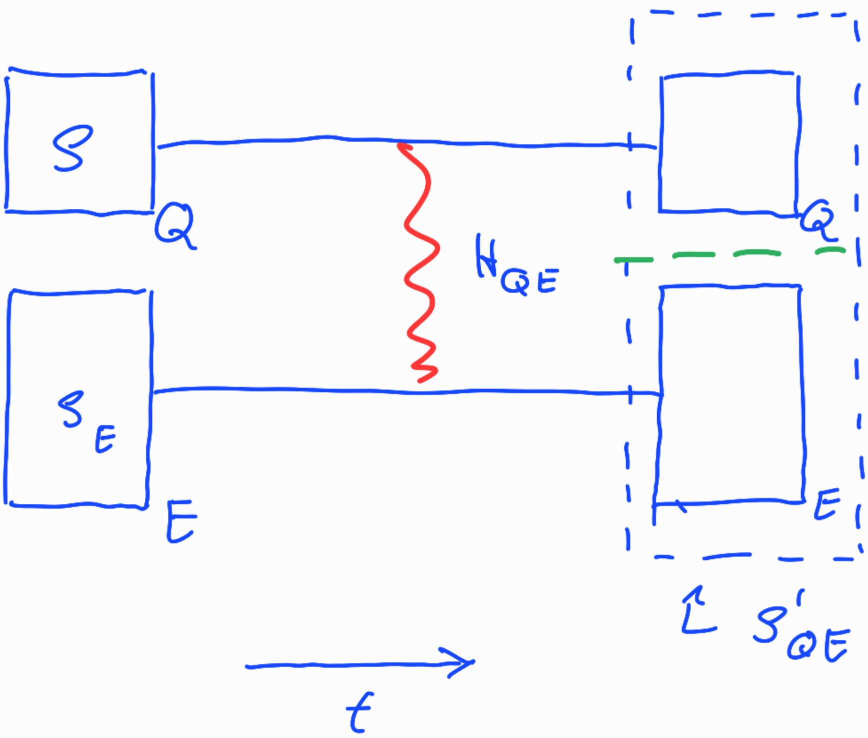
Quantum operation, Kraus-operator expansion

time-evolution of an <u>open</u> quantum system over a time interval $[0,t]$ defines a <u>quantum operation</u> $\mathcal{E}: \rho \mapsto \mathcal{E}(\rho)$,

with

$$\boxed{\mathcal{E}(\rho) = \mathrm{tr}_E \, U \, \rho \otimes \rho_E \, U^\dagger} \quad ;$$

- ρ_E : initial state of environment
- $U = e^{-iH_{QE}t/\hbar}$: time evolution of joint system QE



$\rho' = \mathrm{tr}_E \, \rho'_{QE} =: \mathcal{E}(\rho)$

$\rho'_{QE} = U \rho \otimes \rho_E U^\dagger$

quantum operations can <u>always</u> be

casted in a relatively simple

<u>Kraus-operator expansion:</u>

<u>Thm.:</u>

$\mathcal{E}(\rho)$ quantum operation

$\Longleftrightarrow$

$\exists$ Kraus-ops $E_1, \dots, E_K$ on $\mathcal{H}_Q$ s.t.:

$$\mathcal{E}(\rho) = \sum_{k=1}^{K} E_k \rho E_k^\dagger$$

- $\sum_k E_k^\dagger E_k = \mathbb{1}_Q$
- $K \leq (\dim \mathcal{H}_Q)^2$

<u>Examples:</u>

- unitary time evolution of <u>closed</u> system:

$$\rho \longmapsto \rho_t = U_t \rho U_t^\dagger \quad ; \qquad U^\dagger U = \mathbb{1}$$

$\hat{=}$ Kraus-op. expans. with $K = 1$

- projective measurement:

$$\rho \mapsto \rho' = \sum_{i=1}^{k} P_i \, \rho \, P_i$$

$P_1, \dots, P_n$ orthog. projections, $\sum_{i=1}^{d} P_i = 1$

- <u>bit-flip</u>: $\rho \mapsto \mathcal{E}(\rho) = (1-p)\rho + p\,\sigma_x \rho \sigma_x$

$$= E_1 \rho E_1^\dagger + E_2 \rho E_2^\dagger$$

with $E_1 = \sqrt{1-p}\,\mathbb{1}$, $E_2 = \sqrt{p}\,\sigma_x$.

<u>Proof</u>:

"$\Rightarrow$": w.l.o.g. ρ_E <u>pure</u> state, say

$\rho_E = |\varphi_1\rangle\langle\varphi_1|$ (a mixed state

ρ_E can always be "purified": $\rho_E = \mathrm{tr}_{E'} |\varphi_1'\rangle\langle\varphi_1'|$)

let $|\varphi_1\rangle, \dots, |\varphi_K\rangle$ be ONB of $\mathcal{H}_E$

$$\rightarrow \mathcal{E}(\rho) = \mathrm{tr}_E\, U(\rho \otimes |\varphi_1\rangle\langle\varphi_1|)\, U^\dagger$$

$$= \sum_{k=1}^{k} \sum_{\ell=1}^{k} \langle \varphi_k | U | \varphi_\ell \rangle \underbrace{\langle\varphi_\ell| \, \rho \otimes |\varphi_1\rangle\langle\varphi_1|}_{= \delta_{\ell 1} \rho} U^\dagger |\varphi_k\rangle$$

$$= \sum_{k=1}^{K} \langle \varphi_k | U | \varphi_1 \rangle \, \rho \, \langle \varphi_1 | U^\dagger | \varphi_k \rangle$$

$\rightarrow$ with $E_k := \langle \varphi_k | U | \varphi_1 \rangle$:

$$\mathcal{E}(\rho) = \sum_{k=1}^{K} E_k \rho E_k^{\dagger} .$$

$\ulcorner$ what is meant by $\langle \varphi_k | U | \varphi_1 \rangle$?

with $|1\rangle, \dots, |d\rangle$ ONB of $\mathcal{H}_Q$

$|\varphi_1\rangle, \dots, |\varphi_K\rangle$ ONB of $\mathcal{H}_E$

$$U = \sum_{i,j=1}^{d} \sum_{l,m=1}^{K} u_{ij,\underline{lm}} \, |i\rangle\langle j| \otimes |\varphi_{\underline{l}}\rangle\langle \varphi_{\underline{m}}|$$

$$\rightarrow \langle \varphi_{\underline{k}} | U | \varphi_{\underline{1}} \rangle = \sum_{i,j=1}^{d} u_{ij,\underline{k1}} \, |i\rangle\langle j|$$

<u>operator</u> on $\mathcal{H}_Q$! $\lrcorner$

• $\forall \rho$: $1 \overset{!}{=} \operatorname{tr} \mathcal{E}(\rho) = \operatorname{tr}\left(\sum_k E_k \rho E_k^{\dagger}\right) = \operatorname{tr}\left(\sum_k E_k^{\dagger} E_k\right)\rho$

$$\rightarrow \sum_k E_k^{\dagger} E_k = \mathbb{1}_Q \quad ;$$

• $K \leq (\dim \mathcal{H}_Q)^2$: cf. exercise 19 .

„$\Leftarrow$" given $E_1, \dots, E_K$, $\sum_k E_k^\dagger E_k = \mathbb{1}$, find

$\mathcal{H}_E$, φ_E, and unitary $\mathcal{U}$ on $\mathcal{H}_Q \otimes \mathcal{H}_E$

s.t.
$$\sum_{k=1}^{K} E_k \, \rho \, E_k^\dagger = \operatorname{tr}_E \mathcal{U} \rho \otimes \rho_E \mathcal{U}^\dagger \;!$$
for all ρ

chose $\dim \mathcal{H}_E = K$ with $|\varphi_1\rangle, \dots, |\varphi_K\rangle$ ONB of $\mathcal{H}_E$, $\varphi_E \overset{!}{=} |\varphi_1\rangle\langle\varphi_1|$, and unitary $\mathcal{U}$ s.t.

$$\mathcal{U} \, |i\rangle |\varphi_1\rangle \overset{!}{=} \underbrace{\sum_{k=1}^{K} \left(E_k \, |i\rangle\right) \otimes |\varphi_k\rangle}_{=: \, |\psi_i\rangle}$$

[note: $\langle \psi_i | \psi_j \rangle = \sum_{k,k'=1}^{K} \langle i | E_k^\dagger E_{k'} | j \rangle \langle \varphi_k | \varphi_{k'} \rangle$

$$= \langle i | \underbrace{\sum_{k=1}^{K} E_k^\dagger E_k}_{= \mathbb{1}} | j \rangle = \langle i | j \rangle \quad \checkmark$$
]

then $\quad \mathrm{tr}_E\, U\, \rho \otimes |\varphi_1\rangle\langle\varphi_1|\, U^\dagger$

$$= \sum_{ij} \sum_k \underbrace{\langle\varphi_k| U_E\, |i\rangle|\varphi_1\rangle}_{\hat{=}\, E_k |i\rangle} \langle i|\rho|j\rangle \underbrace{\langle j|\langle\varphi_1|\, U^\dagger\, |\varphi_k\rangle}_{\hat{=}\, \langle j| E_k^\dagger}$$

$$= \sum_k E_k\, \rho\, E_k^\dagger \quad .$$

<u>Unitary freedom in the choice of Kraus operators</u> :

for $\rho_E = |\varphi_1\rangle\langle\varphi_1|$ and ONB $|\varphi_1\rangle, \dots, |\varphi_n\rangle$ of $\mathcal{H}_E$ we have

$$E_k \overset{!}{=} \langle \varphi_k |\, U\, | \varphi_1 \rangle$$

for another ONB $|\chi_1\rangle, \dots, |\chi_n\rangle$ of $\mathcal{H}_E$ we find alternative operators

$$F_e = \langle \chi_e |\, U\, | \varphi_1 \rangle$$

and obtaining

$$\sum_l F_l \, \varsigma \, F_l^\dagger \overset{!}{=} \mathcal{E}(\varsigma) \overset{!}{=} \sum_k E_k \, \varsigma \, E_k^\dagger$$

$\{F_l\}$ and $\{E_k\}$ are related as follows:

$$F_l = \sum_k \underbrace{\langle \chi_l | \varphi_k \rangle}_{=: u_{lk}} \underbrace{\langle \varphi_k | U | \varphi_1 \rangle}_{= E_k} = \sum_k u_{lk} E_k$$

→ any unitary matrix $(u)_{lk}$ transforms $\{E_k\}$ to an equivalent set $\{F_l\}$

with

$$F_l = \sum_k u_{lk} E_k \quad .$$