Quantum operation = completely positive and trace preserving map

general properties of a quantum operation:

(i) linear

(ii) trace preserving

(iii) positive : $\varrho \geq 0 \rightarrow \mathcal{E}(\varrho) \geq 0$

(i) – (iii) sufficient for a map being a proper quantum operation ?

Almost : map needs to be completely positive !

Definitions:

$$\mathcal{U} \in \mathcal{L}(\mathcal{L}(\mathcal{H})), \quad \mathcal{U}' \in \mathcal{L}(\mathcal{L}(\mathcal{H}'))$$

- $\mathcal{U} \otimes \mathcal{U}'(A \otimes B) := \mathcal{U}(A) \otimes \mathcal{U}'(B)$, and by linearity for gen. $\sigma \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}')$

- $A \in \mathcal{L}(\mathcal{H})$ positive $:\Leftrightarrow \forall \psi : \langle \psi | A | \psi \rangle \geq 0$
  ("$A \geq 0$") $\Leftrightarrow A = A^{+}$, eigenv. $\lambda_i \geq 0$.

- $\mathcal{M}$ positive $:\Leftrightarrow$ $\forall A \geq 0 : \mathcal{M}(A) \geq 0$
- $\mathcal{M}$ <u>completely</u> positive

$$:\Longleftrightarrow \quad \forall \mathcal{H}' : \quad \mathcal{M} \otimes \underset{\substack{\uparrow \\ \text{identity on } \mathcal{L}(\mathcal{H}')}}{\mathcal{I}_{\mathcal{H}'}} \text{ positive}$$

<u>Remarks</u>:

- A quantum syst. Q can always be considered as subsystem of a larger system QQ'

$\rightarrow$ not only $\mathcal{E}(\rho_Q)$ needs to be positive, but also $(\mathcal{E} \otimes \mathcal{I}_{Q'})(\rho_{QQ'})$!

- <u>Transposition</u> (w.r.t. to a specific ONB) is a positive map that is <u>not</u> <u>completely</u> positive: e.g. Transpos. T of a qubit state w.r.t. comp. basis:

$$T(|0\rangle\langle 1|) = |1\rangle\langle 0|$$
$$T(|1\rangle\langle 0|) = |0\rangle\langle 1|$$

$\longrightarrow$

for $\varrho_{QQ'} = |\psi\rangle\langle\psi|$ with entangled (!) pure

state vector $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$

$T \otimes I\,(\varrho_{QQ'})$ is <u>not</u> positive:

$$\varrho_{QQ'} = \frac{1}{2}\Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1|$$

$$+ |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \Big)$$

$$\rightarrow T(\varrho_{QQ'}) = \frac{1}{2}\Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1|$$

$$+ |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \Big)$$

$$= \frac{1}{2}\Big(|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10|$$

$$+ |11\rangle\langle 11| \Big)$$

$$\hat{=} \frac{1}{2}\begin{pmatrix} 1 & & & \\ & 0 & 0 & \\ & 1 & 1 & \\ & & & 1 \end{pmatrix}$$ with eigenvalues

$+\frac{1}{2}$ (3 fold)

and $\underline{\underline{-\frac{1}{2}}}$!

$\rightsquigarrow$ $T(\varrho_{QQ'})$ non-positive although $\varrho_{QQ'} \geq 0$!

Thm.:

$\mathcal{E}$ quantum operation

$\Longleftrightarrow$ $\mathcal{E}$ linear, completely positive, trace-preserving map

("$\mathcal{E}$ cptp-map")

Proof:

"$\Rightarrow$": as exercise

"$\Leftarrow$": with $|1\rangle, \dots, |d\rangle$ ONB of $\mathcal{H}$

$|\varphi_1\rangle, \dots, |\varphi_d\rangle$ ONB of same $\mathcal{H}'$

let $|\alpha\rangle := \sum_{i=1}^{d} |i\rangle |\varphi_i\rangle \in \mathcal{H} \otimes \mathcal{H}'$;

$\rightarrow$ by assumption $\sigma := \mathcal{E} \otimes I_{\mathcal{H}'} (|\alpha\rangle\langle\alpha|)$

$= \sum_{ij} \mathcal{E}(|i\rangle\langle j|) \otimes |\varphi_i\rangle\langle\varphi_j|$ is positive

$\rightarrow$ $\sigma = \sum_{l=1}^{d^2} a_l^2 |v_l\rangle\langle v_l|$, $a_l \geq 0$

$\{|v_l\rangle\}$ ONB

map $\mathcal{E}$ can be extracted from state σ:

$$\langle \varphi_i | \sigma | \varphi_j \rangle = \mathcal{E}(|i\rangle\langle j|) \quad !$$

def. anti-linear map $\mathcal{H} \to \mathcal{H}'$ by

$$\mathcal{H} \ni |\psi\rangle = \sum_i \psi_i |i\rangle \quad \longmapsto \quad |\psi^*\rangle = \sum_i \psi_i^* |\varphi_i\rangle \in \mathcal{H}'$$

$$\rightarrow \underline{\langle \psi^* | \sigma | \psi^* \rangle} = \sum_{ij} \psi_i \psi_j^* \langle \varphi_i | \sigma | \varphi_j \rangle$$

$$= \sum_{ij} \psi_i \psi_j^* \mathcal{E}(|i\rangle\langle j|) = \underline{\mathcal{E}(|\psi\rangle\langle\psi|)}$$

define operator A_k on $\mathcal{H}$ by

$$A_k |\psi\rangle := a_k \langle \psi^* | v_k \rangle \qquad \text{(linear!)}$$

$$\rightarrow \mathcal{E}(|\psi\rangle\langle\psi|) = \langle \psi^* | \sigma | \psi^* \rangle = \langle \psi^* | \Big(\sum_k a_k^2 |v_k\rangle\langle v_k| \Big) | \psi^* \rangle$$

$$= \sum_k \underbrace{a_k \langle \psi^* | v_k \rangle}_{=A_k|\psi\rangle} \underbrace{\langle v_k | \psi^* \rangle a_k}_{=\langle\psi| A_k^\dagger}$$

$$= \sum_k A_k |\psi\rangle\langle\psi| A_k^\dagger$$

$$\rightarrow \mathcal{E}(s) \overset{!}{=} \sum_k A_k s A_k^\dagger \; ; \quad \text{i.e. } \mathcal{E} \text{ quantum operation} \quad \blacksquare$$

General aspects of Quantum Error Correction

general problem:

Given a noisy quantum system Q (e.g. n qubits), find subspace $C \subset \mathcal{H}_Q$ s.t. effect of noise on states $|\psi\rangle \in C$ can be corrected!

Some notions:

- noise/ error-model $\hat{=}$ quantum operation $\mathcal{N}$ (on Q)
- subspace $C \subset \mathcal{H}_Q$ $\hat{=}$ quantum code

$$\left(\begin{array}{c} \text{of size } k := \log_2 \dim C \\ \text{and length } n := \log_2 \dim \mathcal{H}_Q \end{array} \right)$$

Definition:

$\mathcal{C}$ is a noise $\mathcal{N}$ correcting code

$:\Leftrightarrow$ $\exists$ error-correcting operation $\mathcal{R}$ on Q

s.t. $\forall\, |\psi\rangle \in \mathcal{C}$

$$\mathcal{R} \circ \mathcal{N}(|\psi\rangle\langle\psi|) \overset{!}{=} |\psi\rangle\langle\psi|$$

$\rightsquigarrow$ Problem:

given noise $\mathcal{N}$ on Q, find $\mathcal{N}$-correcting code $\mathcal{C}$ as large as possible!

an easier problem:

given code $\mathcal{C}$, does it allow to correct noise $\mathcal{N}$?

We gain intuition from the following

Example:

For "unitary noise"

$$\mathcal{N}(S) = \sum_{k=0}^{K} p_k \, U_k \, S \, U_k^{+}$$

unitary op. on $\mathcal{H}_Q$

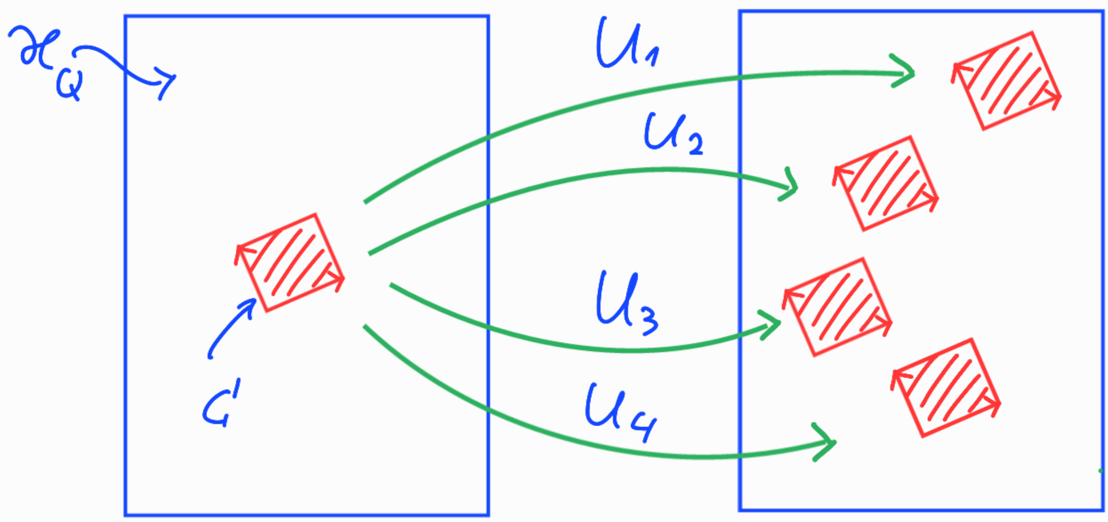
probabilities $\sum p_k = 1$

let code $C \subset \mathcal{H}_Q$ s.t.

$$\boxed{U_k C \perp U_l C} \quad !$$

for all $k \neq l$

pictorially:



→ syndrom measurement by projections

onto $U_h \mathcal{C}$:

$$P_h = U_h \, P \, U_h^\dagger$$

↑ projection on $\mathcal{C}$

→ error correction on syndrom h :

$$U_h^\dagger \quad !$$

→ recovery operation :

$$R(\rho) = \sum_h U_h^\dagger \, P_h \, \rho \, P_h \, U_h$$

$$= \sum_h P \, U_h^\dagger \, \rho \, U_h \, P$$

<u>check</u> : let ρ be a <u>code state</u>, i.e.

$$P \rho P \overset{!}{=} \rho$$

then $R \circ \mathcal{N}(\rho) = \sum_h P \, U_h^\dagger \, \mathcal{N}(\rho) \, U_h \, P$

$$= \sum_h \sum_{g'} n_{g'} \underbrace{P U_h^\dagger U_{g'} P}_{= S_{hg'} P} \overbrace{s}^{\sim = s} \underbrace{P U_{h'}^\dagger U_h P}_{= S_{gg'} P}$$

by the pairwise orthogonality of the transformed code spaces $U_h C$!

$$= \sum_h n_h \, P s P = s \quad \checkmark$$

Note: condition

$$\boxed{P U_{g'}^\dagger U_g P \overset{!}{=} \delta_{g'h} P} ,$$

where P projection on C, ensures that C is a U-correcting code !