Entanglement

- Einstein-Podolsky-Rosen Paradoxon
- Bell-inequalities
- use of entanglement in
  - quantum key distribution
  - quantum teleportation
- Distillation of Bell-pairs from pure-state entanglement
- von Neumann-entropy as entanglement measure
- entanglement of random states

Einstein - Podolsky - Rosen (EPR) Paradoxon

EPR, generally dissatisfied with QM in 1935, devise a "Gedanken"-experiment in order to demonstrate "incompleteness" of QM;

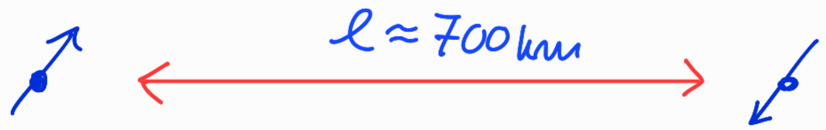
in D. Bohm's 1951 version:

Alice and Bob in distant labs share spins of a pair of spins in entangled state

$$|\psi_{AB}\rangle = \left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \right) / \sqrt{2}$$

Alice

(Aachen)



Bob

(Berlin)

consider two experiments:

(I) at $t=0$ Alice measures Z_A.

QM: result undetermined:

with $p = 1/2$ either $+1$ or -1 ;

according to QM, $|\psi_{AB}\rangle$ as <u>state</u> of the spin pair represents its <u>complete</u> description!

$\rightarrow$ nothing else on earth could tell us more on the actual measurement outcome:

q.m. indeterminism is fundamental!

EPR like to challenge that view by a variant of (I):

(II) again, at $t = 0$ Alice measures Z_A ;

additionally, slightly before at $t' = -\Delta t$, where <u>$\Delta t \ll l/c$</u>, Bob measures Z_B !

<u>QM</u>: Bob's result <u>undetermined</u>: with $p = 1/2$

<u>either</u> $+1$ <u>or</u> -1

$\rightarrow$ state at time $t = -\Delta t + \varepsilon < 0$:

<u>either</u> $|\psi'_{AB}\rangle = -|\downarrow\rangle_A |\uparrow\rangle_B$ <u>or</u> $|\psi'_{AB}\rangle = |\uparrow\rangle_A |\downarrow\rangle_B$

→ Alice's Z_A measurement yields result that seems to be determined by Bob's result:

$$
\begin{array}{lcc}
\text{Bob:} & +1 & -1 \\
 & \downarrow & \downarrow \\
 & |\psi'_{AB}\rangle = -|\downarrow\rangle_A |\uparrow\rangle_B & |\psi'\rangle_{AB} = |\uparrow\rangle_A |\downarrow\rangle_B \\
 & \downarrow & \downarrow \\
\text{Alice} & -1 & +1
\end{array}
$$

<u>EPR</u>: Since $\Delta t \ll \ell/c$, no signal could have told Alice's spin about Bob's measurement result!

→ unless there is some "spooky action at a distance", <u>results</u> of spin measurements must have been <u>determined</u> (by "hidden variables") a long time ($> \ell/c$) before the actual measurements are done; in contradiction to QM, saying that result are undetermined <u>until</u> the spins are measured!

EPR conclude: QM is incomplete, should be replaced by a new theory based on (yet to be identified) local, hidden variables!

SRT-respecting

J. S. Bell 1964: any such theory predicts spin-spin correlations that are constraint by "Bell-inequalities";

– and this can be empirically tested!!

Example: CHSH - inequality

(Clauser, Horne, Shimony, Holt 1969)

Set-up as in EPR paradox:

Alice and Bob in distant lab's perform local measurements on a joint system.

- Alice measures observables Q or R with possible outcomes $q = \pm 1$, $r = \pm 1$
- Bob measures observables S or T with possible outcomes $s = \pm 1$, $t = \pm 1$

$\rightarrow$ $n \gg 1$ measurements on n identically prepared systems are performed and outcomes recorded:

	Measurement A	Measurement B	Outcome a	Outcome b
×	Q	S	$+1$	-1
✓	R	S	-1	$+1$
×	Q	T	-1	-1
×	Q	S	$+1$	$+1$
×	R	T	-1	$+1$
×	Q	S	-1	-1
×	R	T	$+1$	-1

$\rightarrow$ Statistics:

$$\overline{QS} = 1/3$$

$$\overline{RS} = -1$$

$$\overline{QT} = 1$$

$$\overline{RT} = -1$$

Bell, CHSH: any theory based on local, hidden variables predicts:

$$\boxed{\overline{QS} + \overline{RS} + \overline{QT} - \overline{RT} \overset{!}{\leq} 2}$$

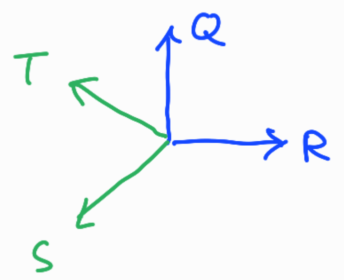
(proof below) $\quad$ CHSH-inequality

QM doesn't hide variables and therefore isn't obliged to make predictions satisfying CHSH-ineq. !

in fact: for $|\psi_{AB}\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$,

$$Q = Z_A \; ; \qquad S = -(Z_B + X_B)/\sqrt{2}$$

$$R = X_A \qquad\quad T = (Z_B - X_B)/\sqrt{2}$$



$$\langle QS\rangle + \langle RS\rangle + \langle QT\rangle - \langle RT\rangle \overset{!}{=} 2\sqrt{2} \underline{\underline{> 2}} \quad !$$

QM violates CHSH-inequality !

<u>Experiments</u> ? (*)

$\boxed{\text{QM is right !}}$

- theories based on local hidden variables are in conflict with reality !

(*) starting with Aspect, Dalibard, Roger 1982

Proof of the CHSH inequality (purely classical!)

Assumption: hidden variables

$$\lambda = (\lambda_1, \lambda_2, \ldots) \in \Gamma$$

give a complete description of system's state and therefore, by use of a suitable (yet to find) theory, predict with certainty measurement outcomes:

$$\Gamma \longrightarrow \{-1, +1\}$$

$$\begin{aligned}
\text{for } Q &: \quad q : \lambda \longmapsto q(\lambda) \\
R &: \quad r : \lambda \longmapsto r(\lambda) \\
S &: \quad s : \lambda \longmapsto s(\lambda) \\
T &: \quad t : \lambda \longmapsto t(\lambda)
\end{aligned}$$

functions $q, r, s, t : \Gamma \to \{-1, +1\}$ are determined by potential hidden variable theory

Note: λ being a local variable means that there are no "actions at a distance"

$\to$ q function of λ and not on e.g. the chosen measurement M_B ($=S$ or T) of Bob!

$$q = q(\lambda) \quad \checkmark \qquad q = \cancel{q(M_B, \lambda)} \;!$$

Whatever the theory is, actual λ-values at time of measurement can be described by probability distribution on Γ:

(as in Stat. Phys!)

$$\boxed{\rho(\lambda)\, d\Gamma} \;\hat{=}\; \text{"State"}$$

normalized: $\displaystyle\int_\Gamma \rho(\lambda)\, d\Gamma = 1$

→ computation of expectation values as

- $$\overline{f(Q)} = \int_\Gamma f(q(\lambda))\, \varrho(\lambda)\, d\Gamma$$

- $$\overline{g(Q,S)} = \int_\Gamma g(q(\lambda), s(\lambda))\, \varrho(\lambda)\, d\Gamma$$

- $$\overline{QS} = \int_\Gamma q(\lambda)\, s(\lambda)\, \varrho(\lambda)\, d\Gamma$$

$\vdots$

→ $$\overline{QS} + \overline{RS} + \overline{QT} - \overline{RT}$$

$$= \int_\Gamma \varrho(\lambda) \Big(\underbrace{\underbrace{qs + rs}_{(q+r)s}}_{} + \underbrace{qt - rt}_{(q-r)t} \Big)_\lambda \, d\Gamma$$

$$\underbrace{(q+r)}_{\pm 2,\ 0}s \qquad \underbrace{(q-r)}_{0,\ \pm 2}t$$

$$\pm 2 \longleftrightarrow 0$$

$$0 \longleftrightarrow \pm 2$$

$$\underbrace{\qquad\qquad\qquad}_{\Sigma\, \leq\, 2}$$

$$\leq 2 \int_\Gamma \varrho(\lambda)\, d\Gamma = 2 \qquad \blacksquare$$