useful in quantum-info (and elsewhere):

Thm.: <u>Schmidt-decomposition</u>

$$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \quad ;$$

$\exists$ orthonormal states $|\varphi_1\rangle, \dots, |\varphi_r\rangle \in \mathcal{H}_A$

orthonormal states $|\chi_1\rangle, \dots, |\chi_r\rangle \in \mathcal{H}_B$

s.t. $$|\psi\rangle = \sum_{i=1}^{r} \lambda_i \, |\varphi_i\rangle |\chi_i\rangle$$

- $\lambda_i \in \mathbb{R}_+$: Schmidt-coefficients
- r : Schmidt rank of $|\psi\rangle$.

$\rightarrow$ • reduced states ρ_A, ρ_B of a general <u>pure</u> state $|\psi_{AB}\rangle$ have <u>identical</u> eigenvalues:

$$|\psi\rangle = \sum_i \lambda_i |\varphi_i\rangle |\chi_i\rangle \quad \xrightarrow{tr_B} \quad \rho_A = \sum_i \lambda_i^2 |\varphi_i\rangle\langle\varphi_i|$$

$$|\psi\rangle = \sum_i \lambda_i |\varphi_i\rangle |\chi_i\rangle \quad \xrightarrow{tr_A} \quad \rho_B = \sum_i \lambda_i^2 |\chi_i\rangle\langle\chi_i|$$

$\rightsquigarrow$ e.g. $S(\rho_A) \overset{!}{=} S(\rho_B)$

- $|\psi_{AB}\rangle$ seperable

$\Leftrightarrow$ $|\psi_{AB}\rangle$ has Schmidt rank 1

$\Leftrightarrow$ $\rho_A = tr_B |\psi_{AB}\rangle\langle\psi_{AB}|$ is a pure state

- Proof of Schmidt-decomp. with Singular Value Decomposition (SVD)

- SVD relies on Polar Decomposition

$\rightarrow$

Polar decomposition

$$\mathcal{L}(\mathcal{H}) \ni A \overset{!}{=} U J \underset{=}{=} K U$$

with positive $J = \sqrt{A^+ A}$, $K = \sqrt{A A^+}$

and unitary U.

proof:
$$A^+ A = \sum_{i=1}^{r} \lambda_i^2 \, |\varphi_i\rangle\langle\varphi_i| \qquad \text{(positive!)}$$

with $\lambda_i \in \mathbb{R}_+$, $r = \operatorname{rank} A$, $|\varphi_1\rangle, \dots, |\varphi_d\rangle$ ONB.

$$\rightarrow \quad J = \sqrt{A^+ A} = \sum_{i=1}^{r} \lambda_i \, |\varphi_i\rangle\langle\varphi_i|$$

$\exists$ another ONB $|\psi_1\rangle, \dots, |\psi_d\rangle$ s.t.

$$|\psi_i\rangle = \frac{1}{\lambda_i} A |\varphi_i\rangle \qquad \text{for } i \leq r$$

$$\ulcorner \; \langle\psi_i|\psi_j\rangle = \frac{1}{\lambda_i \lambda_j} \langle\varphi_i| A^+ A |\varphi_j\rangle = \delta_{ij} \; \lrcorner$$

let unitary U transform

ONB $|\varphi_i\rangle$ to ONB $|\psi_j\rangle$:

$$U = \sum_i |\psi_i\rangle\langle\varphi_i| \quad ,$$

then $\quad UJ \overset{!}{=} A \quad$:

Γ

$i \leq r$: $\quad UJ|\varphi_i\rangle = U\lambda_i|\varphi_i\rangle = A|\varphi_i\rangle$ ✓

$i > r$: $\quad UJ|\varphi_i\rangle = 0 = A|\varphi_i\rangle$ ✓ $\lrcorner$

Furthermore:

$$A = UJ = \underbrace{UJU^\dagger}_{=:K \geq 0} U = KU$$

and $\quad AA^\dagger = KUU^\dagger K = K^2$

$\rightsquigarrow \quad K = \sqrt{AA^\dagger}$

$\boxtimes$

Singular Value Decomposition (SVD)

$$A \overset{!}{=} U D V$$

where U, V are unitary matrices

and D diagonal matrix with

non-negative entries:

$$D = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{pmatrix}$$

$\sigma_i \geq 0$: singular values of A

proof:

$$A = WJ = \overbrace{WX}^{U} D \overbrace{X^{+}}^{V} = UDV$$

Polar Decomp.: W unitary, J positive

$J \geq 0$ can be diagonalized by unitary U:

$J = X D X^{+}$, $d_{ii} \geq 0$

$\blacksquare$

→ Schmidt decomposition

proof:

$$\mathcal{H}_A \otimes \mathcal{H}_B \ni |\psi\rangle = \sum_{i,j=1}^{d} a_{ij} \underbrace{|i\rangle}_{\text{ONB } \mathcal{H}_A} \underbrace{|j\rangle}_{\text{ONB } \mathcal{H}_B}$$

w.l.o.g. $d_A = d_B = d$

S.V.D.: $A = (a_{ij}) \overset{!}{=} U D V$

i.e. $a_{ij} = \sum_{\ell=1}^{r} u_{i\ell} \, d_{\ell\ell} \, v_{\ell j}$, $r = \text{rk } D$

$$\to |\psi\rangle = \sum_{\ell=1}^{r} \underbrace{d_{\ell\ell}}_{=:\lambda_\ell} \underbrace{\sum_i u_{i\ell} |i\rangle}_{=:|\varphi_\ell\rangle} \underbrace{\sum_j v_{\ell j} |j\rangle}_{=:|\chi_\ell\rangle}$$

ONBs because U, V unitary

$$= \underbrace{\sum_{i=1}^{r} \lambda_i \, |\varphi_i\rangle |\chi_i\rangle}_{\text{"Schmidt-form"}}$$

∎

Examples: $\mathcal{H}_A = \mathcal{H}_B = \text{Span}\{|0\rangle, |1\rangle\}$

orthonormal

1) $|\psi_1\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$

$$\overset{!}{=} \underbrace{\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)}_{|\varphi_1\rangle} \otimes \underbrace{\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)}_{|\chi_1\rangle}$$

$\rightarrow$ Schmidt rank $\underline{r = 1}$ ($|\psi_1\rangle$ $\underline{\text{separable}}$)

$\lambda_1 = 1$;

$\rho_A = \text{tr}_B |\psi_1\rangle\langle\psi_1| = |\varphi_1\rangle\langle\varphi_1|$ (pure!)

2) $|\psi_2\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$

$$= \frac{1}{\sqrt{2}} |0\rangle \underbrace{\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)} + \frac{1}{\sqrt{2}} |1\rangle \underbrace{\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)}$$

$$= \frac{1}{\sqrt{2}} |\varphi_1\rangle |\chi_1\rangle + \frac{1}{\sqrt{2}} |\varphi_2\rangle |\chi_2\rangle$$

$\perp$ ✓

$\rightsquigarrow$ Schmidt-rank $r = 2$

i.e. $|\psi_2\rangle$ entangled!

$$\rho_A = \mathrm{tr}_B\, |\psi_2\rangle\langle\psi_2| = \frac{1}{2}\, |\varphi_1\rangle\langle\varphi_1| + \frac{1}{2}\, |\varphi_2\rangle\langle\varphi_2| \; .$$

mixed!