Discrete Fourier transform and its quantum implementation

$f : \mathbb{Z}_N \to \mathbb{C},\ x \mapsto f(x)$ is

transformed to

$F[f] = \hat{f} : \mathbb{Z}_N \to \mathbb{C},\ k \mapsto \hat{f}(k)$

with

$$\boxed{\hat{f}(k) := \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x)\, e^{\frac{2\pi i}{N} x k}}$$

$\leadsto$ unitary n-qubit transformation for DFT on $\mathbb{Z}_N$, $N = 2^n$:

$$\mathcal{F}_n : \mathcal{H}_n \to \mathcal{H}_n ,$$

$$\boxed{\mathcal{F}_n |x\rangle = \frac{1}{N} \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N} x k} |k\rangle}$$

$|x_{n-1} x_{n-2} \cdots x_1 x_0\rangle$ $\qquad$ $|k_{n-1} k_{n-2} \cdots k_0\rangle$

an efficient quantum circuit for F_n follows from

<u>Product decomposition of F_n</u>:

with

- $x = (x_{n-1} x_{n-2} \cdots x_0) = \sum_{\ell=0}^{n-1} x_\ell 2^\ell$
- $0.y_1 y_2 \cdots y_m = \sum_{\ell=1}^{m} y_\ell / 2^\ell$

← binary fraction

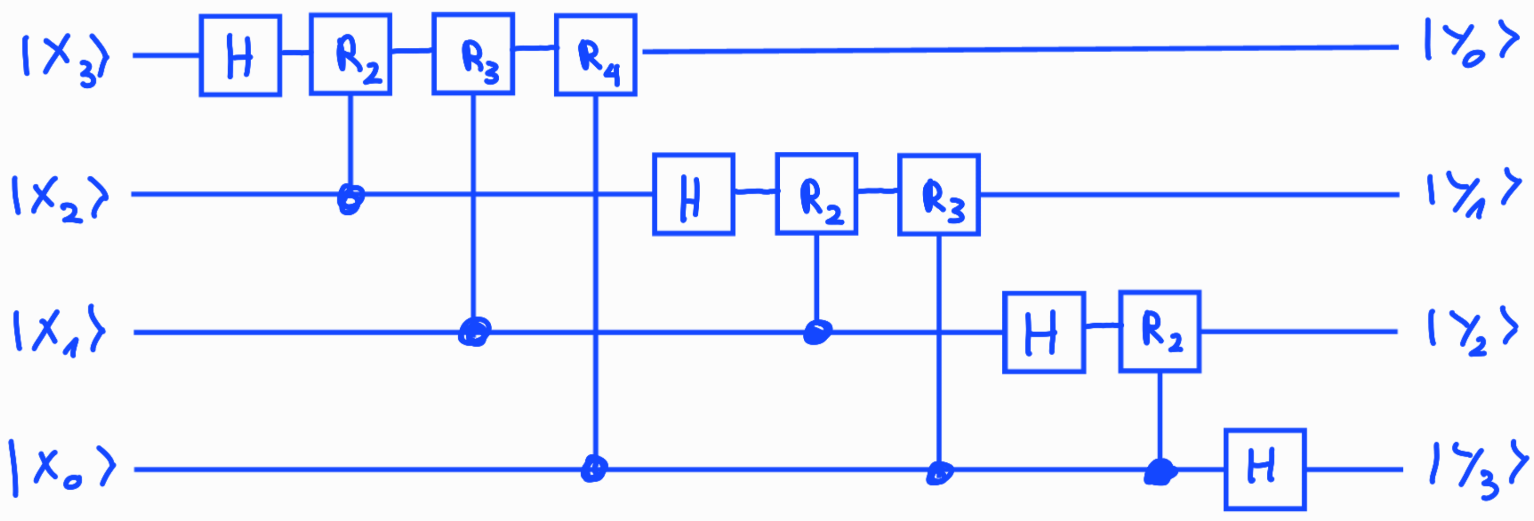
are obtained (see below):

$$F_n |x\rangle \overset{!}{=} \frac{1}{2^{n/2}} \left(|0\rangle + e^{2\pi i\, 0.x_0} |1\rangle \right)$$
$$\otimes \left(|0\rangle + e^{2\pi i\, 0.x_1 x_0} |1\rangle \right)$$
$$\vdots$$
$$(*) \qquad \otimes \left(|0\rangle + e^{2\pi i\, 0.x_{n-1} x_{n-2} \cdots x_1 x_0} |1\rangle \right)$$

⇝ circuit implementation with Hadamard gates $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and controlled phase gates

$$R_l = \begin{pmatrix} 1 & \\ & e^{2\pi i/2^l} \end{pmatrix}$$

as follows: $(n=4)$



$\rightarrow$ circuit for F_n needs

- n Hadamard gates (1-qubit)
- $\frac{1}{2}n(n-1)$ controlled phase gates (2-qubit)

i.e. $O(n^2)$ elementary gates

are sufficient for D.F.T. over $\mathbb{Z}_{2^n}$!

⌈ classical "fast Fourier transform"

needs $O(n \cdot 2^n)$ operations ⌋

Note:

useful application of quantum DFT

requires efficient preparation of

input state $\sum_x f(x) |x\rangle$!

we still have to show the decomp.

of $F_n |x\rangle$ into an n-fold product!

$$2^{n/2}\, \mathcal{F}_n |x\rangle = \sum_{k=0}^{2^n-1} e^{2\pi i x k / 2^n} |k\rangle$$

$$= \sum_{k_{n-1}=0}^{1} \cdots \sum_{k_0=0}^{1} e^{2\pi i x \sum_{\ell=0}^{n-1} k_\ell 2^{\ell-n}} |k_{n-1} \cdots k_0\rangle$$

$$= \bigotimes_{\ell=0}^{n-1} \underbrace{\sum_{k_\ell=0}^{1} e^{2\pi i x k_\ell 2^{\ell-n}} |k_\ell\rangle}_{\parallel}$$

$$|0\rangle + e^{2\pi i \sum_{m=0}^{n-1} x_m 2^{m+\ell-n}} |1\rangle$$

$$= \left(|0\rangle + e^{2\pi i\, 0.x_0} |1\rangle \right) \qquad \ell = n-1$$

$$\otimes \left(|0\rangle + e^{2\pi i\, 0.x_1 x_0} |1\rangle \right) \qquad \ell = n-2$$

$$\otimes \left(|0\rangle + e^{2\pi i\, 0.x_2 x_1 x_0} |1\rangle \right) \qquad \ell = n-3$$

$$\vdots$$

$$\otimes \left(|0\rangle + e^{2\pi i\, 0.x_{n-1} \cdots x_1 x_0} |1\rangle \right) \qquad \ell = 0$$