sheet 1 : Solutions

0. Measurement

a) general $\rho = \sum_i q_i |\varphi_i\rangle\langle\varphi_i|$ may describe ensemble $\mathcal{E} = \{(q_i, |\varphi_i\rangle)\}$. Ideal measurement of A on system in $|\varphi_i\rangle$ yields ensemble $\{ p_\ell^{(i)}, |\varphi_\ell^{(i)}\rangle \}_{\ell=1,\dots,N}$ where $p_\ell^{(i)} = \langle\varphi_i| P_\ell |\varphi_i\rangle$ and $|\varphi_\ell^{(i)}\rangle = \frac{1}{\sqrt{p_\ell^{(i)}}} P_\ell |\varphi_i\rangle$;

$\rightarrow$ initial $\mathcal{E}$ transforms to

$$\mathcal{E}' = \{(\pi_{i\ell} = q_i p_\ell^{(i)}, |\varphi_\ell^{(i)}\rangle)\}_{\substack{i=1,\dots,d \\ \ell=1,\dots,N}}$$

with density operator

$$\rho' = \sum_{i\ell} \pi_{i\ell} |\varphi_\ell^{(i)}\rangle\langle\varphi_\ell^{(i)}|$$

$$= \sum_\ell P_\ell \Big(\underbrace{\sum_i q_i |\varphi_i\rangle\langle\varphi_i|}_{=\rho} \Big) P_\ell = \sum_\ell P_\ell \rho P_\ell .$$

b) w.l.o.g. assume ρ <u>pure</u> (otherwise argue as in a)): $\rho = |\psi\rangle\langle\psi|$;

non-ideal measurement leads to

$$\mathcal{E}' = \{ p_\ell , |\psi_\ell\rangle \}_{\ell = 1,\dots,N}$$

with $p_\ell = \langle\psi|P_\ell|\psi\rangle$ and

$$|\psi_\ell\rangle = \frac{1}{\sqrt{p_\ell}} \underline{V_\ell} P_\ell |\psi\rangle = \frac{1}{\sqrt{p_\ell}} U_\ell |\psi\rangle$$

$$\rightarrow \rho' = \sum_\ell p_\ell |\psi_\ell\rangle\langle\psi_\ell| = \sum_\ell U_\ell |\psi\rangle\langle\psi| U_\ell^\dagger$$

$$= \sum_\ell U_\ell \rho U_\ell^\dagger \quad ;$$

$$\sum_\ell U_\ell^\dagger U_\ell = \sum_\ell P_\ell \underbrace{V^\dagger V}_{= \mathbb{1}} P_\ell = \sum_\ell P_\ell = \mathbb{1} \quad ;$$

$$\mathrm{tr}(U_\ell^\dagger U_\ell \rho) = \mathrm{tr}(P_\ell \underbrace{V^\dagger V}_{= \mathbb{1}} P_\ell |\psi\rangle\langle\psi|)$$

$$= \langle\psi|P_\ell|\psi\rangle = p_\ell \quad .$$

1) here $M_\uparrow = P_\uparrow = |\uparrow\rangle\langle\uparrow|$ and

$M_\downarrow = \sigma_x P_\downarrow = |\uparrow\rangle\langle\downarrow|$,

$$\rightarrow \quad \rho' = M_\uparrow \rho M_\uparrow^+ + M_\downarrow \rho M_\downarrow^+$$

$$= |\uparrow\rangle\langle\uparrow|\rho|\uparrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow|\rho|\downarrow\rangle\langle\uparrow|$$

$$= |\uparrow\rangle \big(\underbrace{\langle\uparrow|\rho|\uparrow\rangle + \langle\downarrow|\rho|\downarrow\rangle}_{\hat{=}\, \mathrm{tr}\,\rho = 1} \big) \langle\uparrow| = \underline{\underline{|\uparrow\rangle\langle\uparrow|}}$$

preparation of "↑"!

1. Criterion for purity

- $\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i| \rightarrow \rho^2 = \sum_i p_i^2 |\varphi_i\rangle\langle\varphi_i|$

$$\rightarrow \mathrm{tr}\,\rho^2 = \sum_i p_i^2 \underset{\substack{\uparrow \\ p_i \geq 0}}{\leq} \Big(\sum_i p_i\Big)^2 = 1$$

- $\underline{\rho \text{ pure}} \rightarrow \rho = |\psi\rangle\langle\psi| \overset{!}{=} \rho^2 \rightarrow \underline{\mathrm{tr}\,\rho^2 = 1}$

• ρ mixed $\longrightarrow$ $\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$

with, say, $p_1 \geq p_2 > 0$

$\rightarrow$ $tr\,\rho^2 = p_1^2 + p_2^2 + \sum_{i>2} p_i^2$

$$\leq p_1^2 + p_2^2 + q^2 \quad , \quad q = \sum_{i>2} p_i \geq 0$$

$$< (p_1 + p_2 + q)^2 = 1 .$$

$p_1, p_2 > 0 \; , \; q \geq 0$

2. Blochsphere

a) $\mathbb{1}, \sigma_1, \sigma_2, \sigma_3$ form a complete system of complex 2×2 matrices.

$\rightarrow$ ρ can be expanded as $\rho = \alpha\mathbb{1} + \frac{1}{2}\vec{r}\cdot\vec{\sigma}$;

since $tr\,\sigma_i = 0$ and $tr\,\rho = 1 \rightarrow \alpha = 1/2$

with $tr\,\sigma_i\sigma_j = 0$, $\sigma_i^2 = \mathbb{1}$, $\vec{r} = (x_1, x_2, x_3)$:

$$tr\,\rho^2 = \frac{1}{4} tr\,(\mathbb{1} + \vec{r}\cdot\vec{\sigma})^2 = \frac{1}{2}(1 + x_1^2 + x_2^2 + x_3^2)$$

$$= \frac{1}{2}(1 + |\vec{r}|^2) \overset{!}{\leq} 1 \rightarrow |\vec{r}| \leq 1 .$$

b)

$|0\rangle: \quad \rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(\mathbb{1} + \sigma_3) \rightarrow \vec{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$|1\rangle: \quad \rho = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(\mathbb{1} - \sigma_3) \rightarrow \vec{r} = -\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\frac{|0\rangle + |1\rangle}{\sqrt{2}}: \quad \rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}(\mathbb{1} + \sigma_1) \rightarrow \vec{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\frac{|0\rangle + i|1\rangle}{\sqrt{2}}: \quad \rho = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = \frac{1}{2}(\mathbb{1} + \sigma_2) \rightarrow \vec{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\rho = \frac{1}{2}\mathbb{1} = \frac{1}{2}(\mathbb{1} + \vec{0}\cdot\vec{\sigma}) \rightarrow \vec{r} = \vec{0}$

c)

ρ pure $\overset{1.}{\Longleftrightarrow}$ $\operatorname{tr}\rho^2 = 1$

$\overset{a)}{\Longleftrightarrow}$ $\frac{1}{2}(1 + |\vec{r}|^2) = 1$

$\Longleftrightarrow$ $|\vec{r}| = 1$.