Exercise 3

We define a family of 2-qubit states $|\psi(\alpha)\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ for $\alpha \in \mathbb{R}$ with $\frac{1}{2} \leq \alpha \leq 1$ as

$$|\psi(\alpha)\rangle = \left(\frac{\alpha}{2}\right)^{\frac{1}{2}} |00\rangle + \left(\frac{\alpha}{2}\right)^{\frac{1}{2}} |11\rangle + \left(\frac{1-\alpha}{2}\right)^{\frac{1}{2}} |01\rangle + \left(\frac{1-\alpha}{2}\right)^{\frac{1}{2}} |10\rangle \tag{1}$$

a) You measure the first qubit. What is the probability of finding the measurement outcome that corresponds to the $|0\rangle$ state for the first qubit?
b) If you indeed find this measurement outcome, what is the post-measurement state?
c) Show that $|\psi(\alpha)\rangle$ is entangled for all values of α, except $\alpha = \frac{1}{2}$. You are allowed to assume that all amplitudes are real.
Hint 1: Assume that $|\psi(\alpha)\rangle$ *is* a product state and work towards a contradiction for values other than $\alpha = \frac{1}{2}$.
Hint 2: Another option is to look at the Schmidt rank of the reduced density matrix $\rho_A = \mathrm{Tr}_B(|\psi(\alpha)\rangle\langle\psi(\alpha)|)$.
Hint 3: A third way is by looking at $\mathrm{Tr}(\rho_A^2)$. If this is your preferred way, also explain why this is sufficient.

Solution

a) Let l be the measurement outcome corresponding to the $|0\rangle$ state of the first qubit.

$$\begin{aligned} P(l) &= \langle\psi(\alpha)| P_0 \otimes \mathbb{I} |\psi(\alpha)\rangle = \sum_x \langle\psi(\alpha)| |0x\rangle\langle 0x| |\psi(\alpha)\rangle \\ &= \langle\psi(\alpha)| |00\rangle\langle 00| |\psi(\alpha)\rangle + \langle\psi(\alpha)| |01\rangle\langle 01| |\psi(\alpha)\rangle \\ &= \frac{\alpha}{2} + \frac{1-\alpha}{2} = \frac{1}{2} \end{aligned}$$

b) To practice the new density matrix formalism, let's do this calculation using $\rho = |\psi(\alpha)\rangle\langle\psi(\alpha)|$ and the measurement rules from exercise 0. The post-measurement state is

$$\begin{aligned} \rho' &= (p(00) + p(01) \frac{(P_{00} + P_{01}) |\psi(\alpha)\rangle\langle\psi(\alpha)| (P_{00} + P_{01})}{\mathrm{Tr}((P_{00} + P_{01}) |\psi(\alpha)\rangle\langle\psi(\alpha)|)} \\ &= \alpha |00\rangle\langle 00| + (1-\alpha) |01\rangle\langle 01| + \sqrt{\alpha(1-\alpha)}(|00\rangle\langle 01| + |01\rangle\langle 00|). \end{aligned}$$

c) There are three (or possibly even more) ways of solving this exercise.
Option 1: Assume $|\psi(\alpha)\rangle$ is a product state of 2 qubits. Then it should be possible to write

$$|\psi(\alpha)\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle),$$

for $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$. Multiplying out, we get

$$|\psi(\alpha)\rangle = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle.$$

So we have $ac = bd = \sqrt{\frac{\alpha}{2}}$ and $ad = bc = \sqrt{\frac{1-\alpha}{2}}$. Substituting $a = bd/c$ into $ad = bc$ we arrive at the conclusion that $c = \pm d$. Similarly, we can derive that $a = \pm b$. Using the normalisation, the only (real) solutions are $a = \pm\frac{1}{\sqrt{2}}$ and the same for b, c, d. I.e. the only valid value for α in the given range is $\alpha = \frac{1}{2}$.
Option 2: Calculate the reduced density matrix ρ_A and determine its rank. This is the Schmidt number of $\psi(\alpha)$.

$$\begin{aligned}
\rho_A &= \mathrm{Tr}_B(|\psi(\alpha)\rangle\langle\psi(\alpha)|) = \sum_i (\mathbb{I} \otimes \langle i|)\,|\psi(\alpha)\rangle\langle\psi(\alpha)|\,(\mathbb{I} \otimes |i\rangle) \\
&= ... = \frac{1}{2}\,|0\rangle\langle 0| + \frac{1}{2}\,|1\rangle\langle 1| + \frac{1}{2}\sqrt{\alpha(1-\alpha)}(|0\rangle\langle 1| + |1\rangle\langle 0|) \\
&= \frac{1}{2}\begin{pmatrix} 1 & \sqrt{\alpha(1-\alpha)} \\ \sqrt{\alpha(1-\alpha)} & 1 \end{pmatrix}
\end{aligned}$$

This matrix is a rank 1 matrix if and only if $\alpha = \frac{1}{2}$ if α is restricted to the reals. If the Schmidt rank is larger than 1, the state is entangled.
Option 3: Again looking at ρ_A, calculate

$$\begin{aligned}
\mathrm{Tr}\big(\rho_A^2\big) &= \mathrm{Tr}\left(\frac{1}{4}\begin{pmatrix} 1 + 4\alpha(1-\alpha) & \dots \\ \dots & 1 + 4\alpha(1-\alpha) \end{pmatrix}\right) \\
&= \frac{1}{4} \cdot 2 \cdot (1 + 4\alpha(1-\alpha)) = \frac{1}{2} + 2\alpha(1-\alpha).
\end{aligned}$$

This equals 1 if and only if $\alpha = \frac{1}{2}$. This is a sufficient condition, because it means that ρ_A is pure and pure states can't be entangled with other systems. Conversely, every mixed state is entangled with its purification.