Solutions problem sheet 3

9a) $\dim W$ increases by 1 iff $\ell_{k+1} \notin W$;

this happens with prob.

$$p = \frac{2^{\dim V} - 2^{\dim W}}{2^{\dim V}}$$

$$= 1 - 2^{\underbrace{\dim W - \dim V}_{\leq -1}} \geq 1/2 .$$

9b) If $\dim W$ increases by 1 with

$p = 1/2$ per random drawing of $\ell \in V$,

after K drawings

$$\mathrm{Prob}(\dim = d) = \binom{k}{d} p^d (1-p)^{k-d}$$

$$= \frac{1}{2^k} \binom{k}{d} .$$

9e) the binomial distribution

$$p_n(d) = \frac{1}{2^n} \binom{n}{d}$$

has mean $\mu = p\,n = n/2$ and

variance $\sigma^2 = k\,p(1-p) = n/4$

$\rightarrow$ for large k approximation by gaussian

$$P_n(d) = \frac{1}{\sqrt{\pi n/2}}\, e^{-2(d-k/2)^2/k}$$

$$\rightarrow \quad P_{4n}(n-2) \approx P_{4n}(n) \sim e^{-n/2},$$

$$\rightarrow \quad \mathrm{Prob}(d \leq n-2) \leq (n-2)\, P_{4n}(n-2) \sim e^{-n/2}.$$

10a) We show that the hermitian

adjoint F_N^+, given by

$$F_N^+ |k\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} e^{\overset{!}{-}\frac{2\pi i}{N} x k} |x\rangle ,$$

actually is the inverse of F_N:

$$F_N^+ F_N |x_0\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i x_0 k/N} F_N^+ |k\rangle$$

$$= \sum_{x=0}^{N-1} \underbrace{\frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i (x_0 - x) k/N}}_{=: A_{x_0,x} \overset{!}{=} \delta_{x_0,x}} |x\rangle = |x_0\rangle \quad \checkmark$$

$\ulcorner$

$$x = x_0 : \quad A_{x_0,x_0} = \frac{1}{N} \sum_{k=0}^{N-1} 1 = 1 \quad \checkmark$$

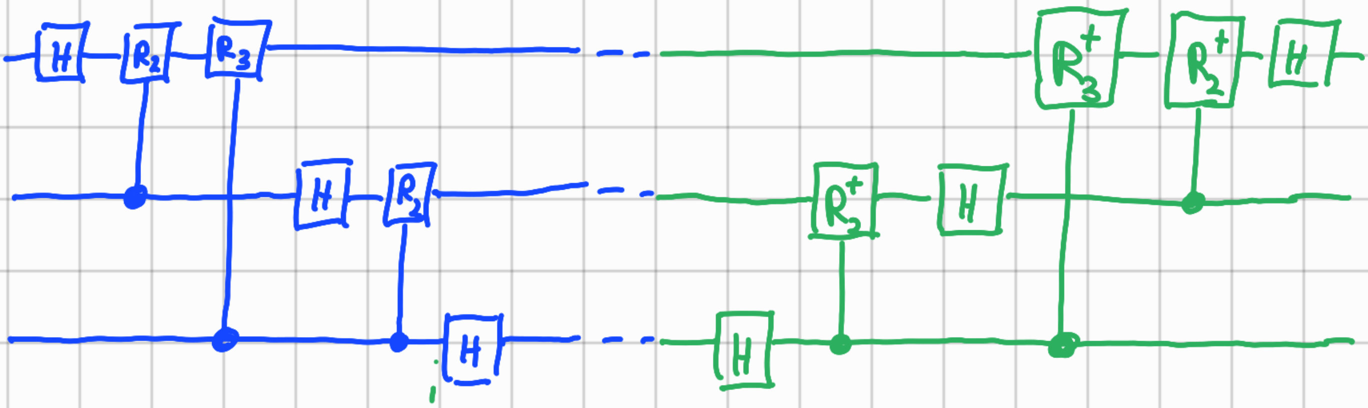
$x_0 - x = m \neq 0$:

$$A_{x_0,x} = \frac{1}{N} \sum_{k=0}^{N-1} \left(e^{2\pi i \frac{m}{N}}\right)^k$$

$$= \frac{1}{N} \frac{1 - e^{2\pi i m}}{1 - e^{2\pi i \frac{m}{N}}} = 0 \quad \checkmark$$

$\lrcorner$

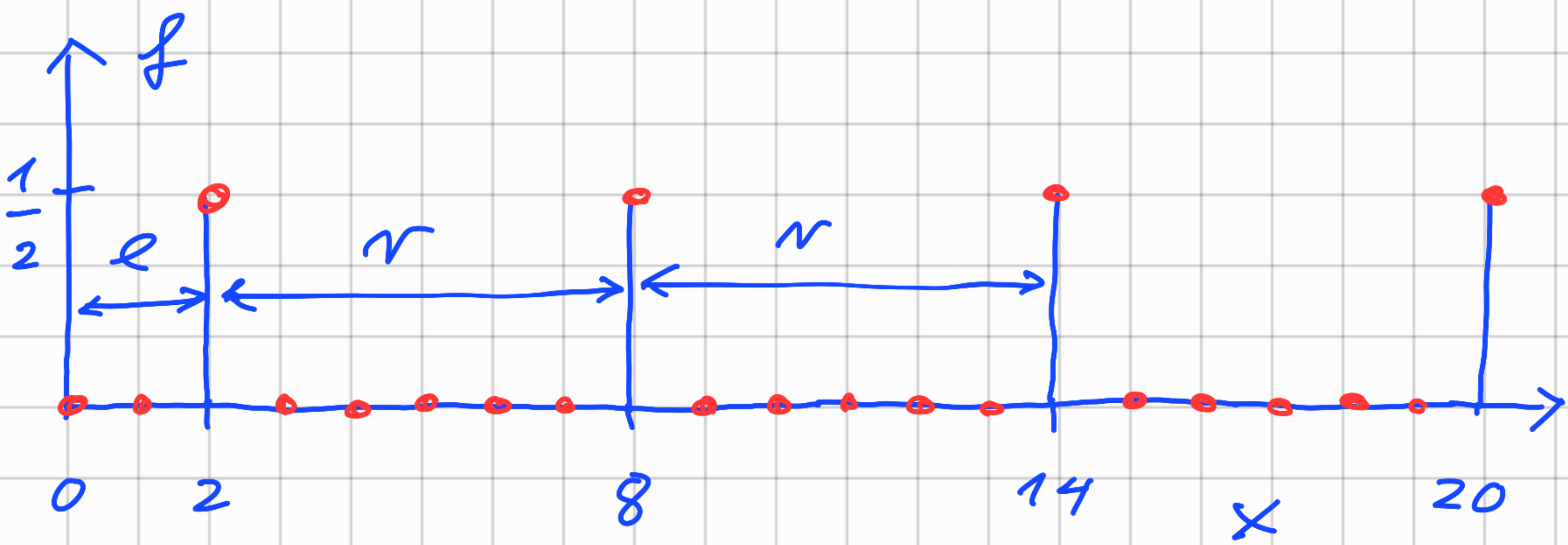
10 b) for simplicity $n = 3$:



- the l.h. circuit implements $\underline{\underline{F_3}}$ (lecture!)
- the entire circuit obviously implements $\underline{\underline{\mathbb{1}_3}}$

$\rightarrow$ the green r.h. circuit implements $\underline{\underline{F_3^{-1}}}$!

11 a)



11 b)

$$\hat{f}(h) = \frac{\sqrt{r}}{N} \sum_{j=0}^{N/r - 1} e^{2\pi i (l + jr) h / N}$$

$$= \frac{\sqrt{r}}{N} e^{2\pi i l h / N} \underbrace{\sum_{j=0}^{N/r-1} \left(e^{2\pi i \frac{r h}{N}} \right)^{j}}_{=: B_h}$$

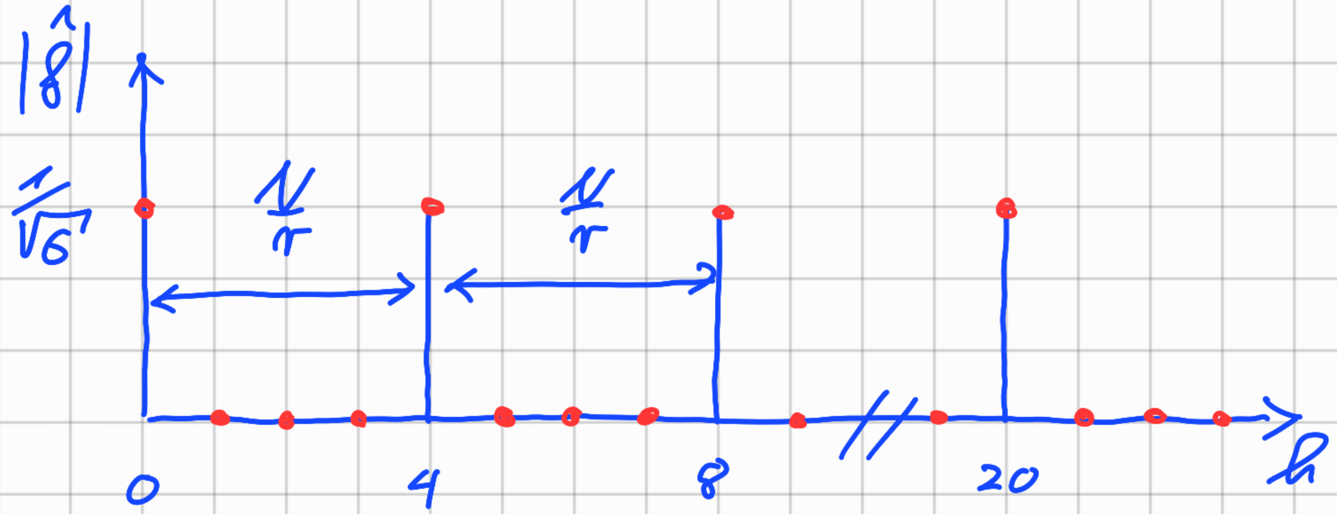
same calc. as in 10a) (or lecture) shows:

$$B_h = \begin{cases} N/r & : \; rh = mN \quad \text{for } m = 0, \dots, r-1 \\ 0 & : \; \text{else} \end{cases}$$

hence:

$$\hat{f}(h) = \frac{e^{2\pi i l h / N}}{\sqrt{r}} \begin{cases} 1 & : \; h = 0 \mod \frac{N}{r} \\ 0 & : \; h \neq 0 \mod \frac{N}{r} \end{cases}$$

1) $\frac{N}{r} = \frac{24}{6} = 4$

2) 

d)

- for $r=1$, $\ell = 0$ with b) : $\hat{f}(k) = 1$.

- for $r=2$, $\ell = 0$, N even with b) :

$$\hat{g}(k) = \frac{1}{\sqrt{2}} \begin{cases} 1 : & k = 0 \bmod \frac{N}{2} \\ 0 : & k \neq 0 \bmod \frac{N}{2} \end{cases}$$

i.e. $\hat{g}(0) = \hat{g}\left(\frac{N}{2}\right) = 1/\sqrt{2}$;

for $k \neq 0, \frac{N}{2}$: $\hat{g}(k) = 0$.