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Quantum Information Theory – Sheet 1

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Webpage: *http://www.thp.uni-koeln.de/∼rk/qit_22.html/*

Submission of solutions as pdf-file until Thursday, April 21, 12 pm, to
ligthart.exams[at]gmail.com
Joint solutions of teams with up to three members are allowed, welcomed, and highly recommended!

0. Measurement

2+2+2=6 Punkte

a) We consider an ideal measurement of the observable $A = \sum_{l=1}^{N} a_l P_l$ (a_l are the eigenvalues of A, P_l the projections on the eigenspaces) of a quantum system, which initially is in a ***general*** state ρ. Show that, if the measurement outcome is not known, the state of the system immediately after the measurement is

$$\rho' = \sum_{l=1}^{N} P_l \rho P_l \, .$$

b) Now we consider a *non-ideal* measurement, in which an outcome a_l triggers a unitary transformation V_l of the system's state. Again, assume the measurement outcome is not known and show that in this case the final state is

$$\rho' = \sum_{l=1}^{N} M_l \rho M_l^+ \, ,$$

where $M_l = V_l P_l$. Verify

$$\sum_{l=1}^{N} M_l^+ M_l = \mathbf{1}, \qquad p_l = \mathrm{Tr}(M_l^+ M_l \rho) \, ,$$

where p_l is the probability of measurement outcome a_l.

c) Specifically, we consider a non-ideal spin-z measurement

$$\sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

on a spin-$1/2$ particle. In this measurement the outcome "$\downarrow$" triggers

$$\sigma_x = |\downarrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \, ,$$

while the outcome "$\uparrow$" leaves the system unchanged. What is the final state ρ' on measuring a general initial state ρ ?

1. A criterion for purity *4 Punkte*

Let ρ be a density operator. Show that $\mathrm{Tr}(\rho^2) \leq 1$, with equality if and only if ρ is a pure state.

2. Blochsphere *3+3+3=9 Punkte*

Let ρ be the density operator of a general qubit state. The operator ρ represented as 2×2 matrix w.r.t. to the orthonormal system $\{|0\rangle, |1\rangle\}$ then defines the *density matrix* $\hat{\rho}$ of that state.

a) Show that $\hat{\rho}$ can be written as

$$\hat{\rho} = \frac{1}{2} \left(\mathbf{1} + \vec{r} \cdot \vec{\sigma} \right) ,$$

where the *Bloch vector* $\vec{r}$ is a real three-dimensional vector such that $|\vec{r}| \leq 1$. $\vec{\sigma} = (\sigma_2, \sigma_2, \sigma_3)$ denote the standard Pauli matrices.

b) What are the Bloch vectors corresponding to pure qubit states $|0\rangle$, $|1\rangle$, $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle+i|1\rangle}{\sqrt{2}}$? What is the Bloch vector representing the state $\rho = \frac{1}{2}\mathbf{1}$?

c) Show that ρ is pure if and only if $|\vec{r}| = 1$. **Hint:** Problem 1.

3. Entanglement *2+2+6=10 Punkte*

We define a family of 2-qubit states $|\psi(\alpha)\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ for $\alpha \in \mathbb{R}$ with $\frac{1}{2} \leq \alpha \leq 1$ as

$$|\psi(\alpha)\rangle = \left(\frac{\alpha}{2}\right)^{\frac{1}{2}} |00\rangle + \left(\frac{\alpha}{2}\right)^{\frac{1}{2}} |11\rangle + \left(\frac{1-\alpha}{2}\right)^{\frac{1}{2}} |01\rangle + \left(\frac{1-\alpha}{2}\right)^{\frac{1}{2}} |10\rangle$$

a) You measure the first qubit. What is the probability of finding the measurement outcome that corresponds to the $|0\rangle$ state for the first qubit?

b) If you indeed find this measurement outcome, what is the post-measurement state?

c) Show that $|\psi(\alpha)\rangle$ is entangled for all values of α, except $\alpha = \frac{1}{2}$. You are allowed to assume that all amplitudes are real.
Hint 1: Assume that $|\psi(\alpha)\rangle$ *is* a product state and work towards a contradiction for values other than $\alpha = \frac{1}{2}$.
Hint 2: Another option is to look at the Schmidt rank of $|\psi(\alpha)\rangle$.
Hint 3: A third way is by looking at $\mathrm{Tr}(\rho_A^2)$. If this is your preferred way, also explain why this is sufficient.