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Quantum Information Theory – Sheet 2

Wintersemester 2021/22

Webpage: *http://www.thp.uni-koeln.de/∼rk/qit_22.html/*

Submission of solutions as pdf-file until Thursday, May 5, 12 pm, to
ligthart.exams[at]gmail.com
Note: Out of the **five** exercises of this sheet you need to submit **solutions for four exercises only**.

4. Purifications *6 points*

Suppose a state ρ_A of a system A is purified by a pure state $|\psi_1\rangle$ of a joint system AB and at the same time also by a pure state $|\psi_2\rangle$ of AB. I.e.

$$\mathrm{Tr}_B\,|\psi_1\rangle\langle\psi_1| \;=\; \rho_A \;=\; \mathrm{Tr}_B\,|\psi_2\rangle\langle\psi_2|\,.$$

Show that then there is a unitary transformation U_B on B such that

$$|\psi_1\rangle = (\mathbf{1}_A \otimes U_B)|\psi_2\rangle\,.$$

Hint: consider $|\psi_1\rangle$ and $|\psi_2\rangle$ in *Schmidt-form*.

5. Measurements on the other systems don't matter *6 points*

We consider a joint system AB in a common state ρ_{AB}, meaning that subsystem A is in the reduced state $\rho_A = \mathrm{Tr}_B\,\rho_{AB}$. Does this state change if subsystem B is measured? Generally, such a measurement transforms ρ_{AB} to

$$\rho'_{AB} = \sum_l (\mathbf{1}_A \otimes M_l)\rho_{AB}(\mathbf{1}_A \otimes M_l^+)$$

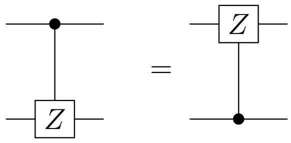
where the measurement operators M_l satisfy $\sum_l M_l^+ M_l = \mathbf{1}_B$ (c.f. Problem 0.). It is assumed that the outcomes are not used to select states. Show that the measurement on B does not change the reduced state in A, i.e.

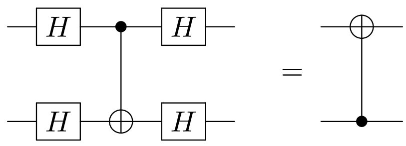
$$\rho_A \stackrel{!}{=} \rho'_A \;=\; \mathrm{Tr}_B\,\rho'_{AB}\,.$$

Hint: show first that $\mathrm{Tr}_B(\mathbf{1}_A \otimes B)O_{AB} = \mathrm{Tr}_B\,O_{AB}(\mathbf{1}_A \otimes B)$.

6. Equivalent circuits *6 points*

Show the following two circuit identities:



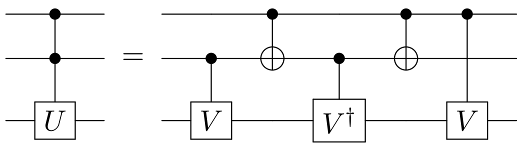


7. Classical and quantum CCNOT

2+2+2+0=6 points

a) Show that the CCNOT-gate is a *universal* classical gate.
Hint: By the universality of {NAND, BRANCH} it suffice to show that NAND and BRANCH gates can be implemented with circuits composed of CCNOT gates only.

b) Verify the following identity of *quantum* circuits:



The left hand circuit is a doubly controlled unitary U. In the right hand circuit V is any unitary satisfying $V^2 = U$.

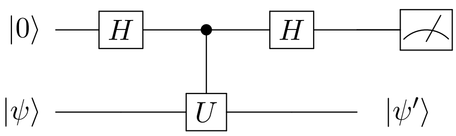
c) Use the above circuit identity in order to find a quantum circuit that implements CCNOT in terms of 2-qubit gates.

d) Argue by the results of **a)** and **c)** that any *classical* computation can be done with a quantum circuit that uses at most 2-qubit-gates.

8. Circuit for an ideal measurement

6 points

Suppose U is a unitary operator on a qubit with eigenvalues -1 and $+1$. This means that U is also hermitian and thus represents a qubit observable. Consider the following circuit:



Show that on measurement outcome 0 the state $|\psi'\rangle$ is the eigenstate of U for eigenvalue $+1$ and on outcome 1 $|\psi'\rangle$ is the eigenstate for eigenvalue -1. The initial state $|\psi\rangle$ is an unspecified pure qubit state.